Hamiltonian Theory: Dynamics

哈密顿理论: 动力学

Thomas Thiemann and Kristina Giesel

托马斯·蒂曼克里斯蒂娜·吉塞尔

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T. Thiemann (□) · K . Giesel

T. 蒂曼 (□) · K . 吉泽尔

Department of Physics, Institute for Theoretical Physics III, Friedrich-Alexander Universität Erlangen-Nürnberg, Erlangen, Germany

德国埃尔兰根弗里德里希-亚历山大埃尔兰根-纽伦堡大学理论物理第三研究所物理系

e-mail: thomas.thiemann@fau.de; thomas.thiemann@gravity.fau.de; kristina.giesel@fau.de; kristina.giesel@gravity.fau.de

电子邮箱:thomas.thiemann@fau.de; thomas.thiemann@gravity.fau.de; kristina.giesel@fau.de; kristina.giesel@gravity.fau.de

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This chapter focuses on the status of the implementation of the dynamics in the canonical version of loop quantum gravity (LQG). Concretely, this means to provide a mathematical meaning of the quantum Einstein equations, sometimes called Wheeler-DeWitt equations, to give a physical interpretation and Hilbert space structure to its solutions, and to construct a representation of the algebra of observables including a physical Hamiltonian. This is a structural overview intentionally skipping technical details.

本章聚焦圈量子引力 (LQG) 正则形式中动力学实现的研究现状。具体而言，我们要为量子爱因斯坦方程 (有时也称为惠勒-德维特方程) 赋予数学含义，为其解给出物理解释并构建希尔伯特空间结构，同时构造包含物理哈密顿量在内的可观测量代数表示。本文为结构性综述，有意略去了技术细节。

Keywords

关键词

Canonical quantum gravity - Hamiltonian constraint - Wheeler-DeWitt equations · Dirac quantization · Reduced phase space quantization · Observables - Physical Hamiltonian

正则量子引力-哈密顿约束-惠勒-德维特方程 · 狄拉克量子化 · 约化相空间量子化 · 可观测量-物理哈密顿量

Introduction

引言

The canonical or Hamiltonian approach to quantum gravity has a long tradition [32-35, 127]. The Hamiltonian formulation of classical general relativity (GR) is also the mathematical framework that underlies its initial value formulation [126] in the globally hyperbolic setting and thus is at the heart of numerical GR [79] in particular in its application to gravitational radiation [31, 81]. A key role is played by the initial value constraints, known as the spatial diffeomorphism D and Hamiltonian constraints C , respectively. These appear in the canonical formulation of any generally covariant field theory [67].

正则或哈密顿量子引力方法有着悠久的研究传统 [32-35, 127]。经典广义相对论 (GR) 的哈密顿表述也是整体双曲背景下其初值公式 [126] 的基础数学框架，因此处于数值广义相对论 [79] 的核心，尤其是在引力辐射的应用方面 [31, 81]。其中初值约束发挥着关键作用，分别被称为空间微分同胚 D 和哈密顿约束 C 。这些约束出现在任意广义协变场论的正则表述中 [67]。

The physical interpretation of those constraints is as follows: In the globally hyperbolic setting, the spacetime manifold M is foliated by a one-parameter family of spacelike hypersurfaces $t \mapsto \sum_t$. The spacetime metric g can be pulled back to those, resulting in the metric $q(t)$ intrinsic to it. On the other hand, the extrinsic curvature $K(t)$ of those hypersurfaces provides information about how that intrinsic metric changes with respect to the ”time” label t of the foliation. Together, the pair $(q(t), K(t))$ is a complete initial data set on $\sum_t$ coordinatizing the phase space of GR. The initial value constraints are functionals of those, and their Hamiltonian flow can be interpreted as spacetime diffeomorphisms tangential and transversal to the foliation. In particular, those flows together with the requirement that the constraints have to vanish are in 1-1 correspondence with the Lagrangian Einstein equations.

这些约束的物理解释如下: 在整体双曲背景下，时空流形 M 被单参数族类空超曲面 $t \mapsto \sum_t$ 叶化。时空度规 g 可以拉回到这些超曲面上，得到超曲面上的内蕴度规 $q(t)$ 。另一方面，这些超曲面的外曲率 $K(t)$ 提供了内蕴度量如何随叶化的" 时间" 参数 t 变化的信息。二者组成的对 $(q(t), K(t))$ 就是 $\sum_t$ 上的完备初始数据集，对应了广义相对论的相空间坐标。初值约束是该数据集的泛函，它们的哈密顿流可以解释为切于和横截于叶化的时空微分同胚。特别是，这些流加上约束必须为零的要求，与拉格朗日形式的爱因斯坦方程是一一对应的。

Although this brings canonical GR very close in appearance to the Hamiltonian formulation of an ordinary field theory, a conceptual difficulty is implied by the fact that the evolution with respect to the label t is considered a gauge transformation, namely, a "temporal diffeomorphism," or coordinate transformation (at least when the equations of motion hold) which therefore is not observable. Moreover, GR is a field theory whose "Hamiltonian" is constrained to vanish as the canonical generator of the flow is a linear combination of the constraints D, C and observables, that is, gauge-invariant quantities, are "constants of motion" with respect to this "Hamiltonian," i.e., they have vanishing Poisson brackets with all constraints when the constraints hold. This set of conceptual differences with respect to an ordinary field theory (say Maxwell theory on Minkowski space) is often called "problem of time." Solving these conceptual problems is therefore part of the answer to the question of how to address the quantum dynamics of GR.

尽管这使得正则广义相对论在形式上非常接近普通场论的哈密顿表述，但一个概念困难来源于: 对参数 t 的演化被认为是一种规范变换，即" 时间微分同胚"，也就是坐标变换 (至少在运动方程成立时)，因此它是不可观测的。此外，广义相对论是这样一种场论: 它的" 哈密顿量" 被约束为零，因为流的正则生成元是约束 D, C 的线性组合，而可观测量 (即规范不变量) 相对于这个" 哈密顿量" 是" 运动常数"，也就是说当约束满足时，它们和所有约束的泊松括号都为零。这套相对于普通场论 (例如闵氏空间上的麦克斯韦理论) 的概念差异通常被称为" 时间问题"。解决这些概念问题因此是解答如何处理广义相对论量子动力学问题的一部分。

Another peculiar aspect of those constraints is their closure, i.e., their evolution with respect to each other or, in other words, their stability under gauge transformations. It turns out that the corresponding Poisson algebra of constraints has a universal structure [67] independent of the concrete generally covariant field theory in question which is called hypersurface deformation algebra $\mathfrak{h}$ and its exponentiation, the "Bergmann-Komar group" $ [20, 21] . In the strict technical sense, $\mathfrak{h}$ is not a sub-Lie algebra of the Poisson algebra of smooth functions on the phase space (and correspondingly $\mathfrak{H}$ is not a Lie group) but something more general that sometimes is called an "algebroid" or "open algebra." What this means is that the Poisson bracket of constraints is a linear combination of constraints; however, unlike the case of a true Lie algebra, the coefficients in that linear combination are non-constant rather than constant functions on the phase space, depending on the inverse of q . In the classical theory, this has no further consequences; it just shows that if the initial value constraints hold on an initial hypersurface, they will hold on all of them, at least as long as q is non-degenerate which is part of the definition of the globally hyperbolic setting. However, in the quantum theory, one expects difficulties in addition to those that arise when trying to represent a classical Poisson Lie algebra by operators on a Hilbert space without anomalies. In particular, as pointed out in [64], although the constraints are real-valued classical functions, they must not be represented by symmetric or even self-adjoint operators. As the constraints encode a local gauge symmetry, anomalies are disastrous as they can indicate that the physical Hilbert space, which by definition is the (generalized) kernel of the quantum constraints, encodes fewer degrees of freedom than the classical theory has. A proper implementation of the quantum constraints therefore has to address their anomaly freeness squarely.

这些约束的另一个特殊性质是它们的封闭性，即约束相对于彼此演化，换句话说，约束在规范变换下保持不变。事实证明，对应的约束泊松代数具有通用结构 [67]，不依赖于所研究的具体广义协变场论，该结构被称为超曲面形变代数 $\mathfrak{h}$，而它的指数化形式则是“贝格曼-科马尔群” $\mathfrak{H}$ [20, 21] 。从严格技术意义上讲，$\mathfrak{h}$ 不是相空间上光滑函数泊松代数的子李代数 (相应地 $\mathfrak{H}$ 也不是李群)，而是一种更一般的结构，有时被称为“代数胚”或“开代数”。这意味着约束的泊松括号是约束的线性组合；但和真正李代数的情况不同，该线性组合的系数不是相空间上的常数函数，而是非常数函数，取决于 q 的逆。在经典理论中，这不会带来进一步的影响；它只是说明，如果初值约束在初始超曲面上成立，那么只要 q 非退化 (这是整体双曲背景定义的一部分)，约束就会在所有超曲面上成立。然而在量子理论中, 除了尝试用希尔伯特空间上的算符无反常地表示经典泊松李代数时已经会遇到的困难之外，我们还会遇到额外的困难。具体而言，正如文献 [64] 指出的，尽管约束是实值经典函数，但它们不能由对称甚至自伴算符表示。由于约束编码了局部规范对称性，反常会带来灾难性后果: 它们表明，按定义是量子约束 (广义) 核的物理希尔伯特空间，所编码的自由度比经典理论更少。因此，要恰当实现量子约束，必须直面解决无反常性的问题。

On the mathematical side, a major difficulty arises due to the tremendous degree of non-linearity of GR. In an ordinary field theory, the Hamiltonian is usually a polynomial in the canonical coordinate functions (here, (q, K)), and the main problem consists in giving meaning to the part of the polynomial beyond quadratic order. The quadratic part typically singles out a certain Fock representation, and the higher-order part, even when normal ordered, is ill-defined as products of operator-valued distributions are involved. In fortunate cases, these ill-defined expressions can be tamed using perturbative renormalization of the S matrix. However, in quantum gravity, the situation is much worse because the ”Hamiltonian” is no longer polynomial. Again, this cannot be avoided in a generally covariant theory because for reasons of gauge or general coordinate invariance, only the integral of scalar densities has a meaning which typically involves inverse and non-integral powers of the determinant of the metric. This is the reason why GR is not renormalizable [62,63], as stated in Hamiltonian terms. Therefore, in quantum gravity, already the proper mathematical definition of the objects that determine the theory presents a major difficulty which must be addressed before reliable physical predictions can be made. Since GR appears not to be perturbatively renormalizable as a standard QFT on Minkowski space in terms of its perturbations (gravitons) of Minkowski space, it is believed that QG must be formulated non-perturbatively and background independently.

在数学层面，广义相对论的高度非线性带来了一个主要困难。在普通场论中，哈密顿量通常是正则坐标函数的多项式 (此处正则坐标就是 (q, K))，主要问题在于为多项式中二次阶以上的部分赋予定义。二次项通常可以筛选出特定的福克表示，而高阶项即使经过正规序处理，仍然是不适定的，因为其中涉及算符值分布的乘积。在比较理想的情况下，可以利用 S 矩阵的微扰重整化来处理这些不适定的表达式。但在量子引力中，情况要糟糕得多，因为“哈密顿量”不再是多项式。这一点在广义协变理论中同样无法避免，因为出于规范或广义坐标不变性的原因，只有标量密度的积分才有良好定义，而这类积分通常涉及度规行列式的逆次幂和非整数次幂。用哈密顿框架的语言来说，这就是广义相对论不可重整 [62,63] 的原因。因此，在量子引力中，甚至对决定该理论的对象给出恰当的数学定义都是一个主要困难，必须先解决这个问题，才能得到可靠的物理预言。由于广义相对论作为闵氏空间上的标准量子场论，就其对闵氏空间的微扰 (引力子) 而言似乎不是微扰可重整的，因此人们认为量子引力必须以非微扰、不依赖背景的方式表述。

In this chapter, we take a snapshot of the status of the particular incarnation of the canonical approach to quantum gravity coined loop quantum gravity (LQG) [45, 87, 108] which is manifestly background independent and non-perturbative. The name is due to the fact that one uses a classically equivalent reformulation of GR in terms of connections rather than metrics and therefore one can take advantage of the arsenal of techniques developed for such quantum gauge theories, in particular gauge-covariant Wilson loop variables [30]. The focus will be on providing an overview about the different strategies that have been invented to address the issues mentioned above and to inform about their respective advantages, disadvantages, and stages of development. In section ”Survey of LQG Dynamics”, we begin with an overview or roadmap over the different developments of the past three decades. We try to be as non-technical as possible. There are two major routes, the quantum constraint approach and the reduced phase space approach which constrain the theory after or before quantization, respectively. These two approaches are typically semiclassically equivalent but may differ with respect to the quantum corrections, and thus, the corresponding phenomenology is affected by this choice. In section ”Constraining After Quantization”, we detail out the quantum constraint approach. In section ”Constraining Before Quantization”, we detail out the reduced phase space approach. In section ”Summary and Outlook”, we summarize and conclude.

本章我们将概述被称为圈量子引力 (LQG) [45, 87, 108] 的正则量子引力方案的现状，该方案是显式背景无关的非微扰理论。该命名源于它采用了以联络而非度规表述的经典等效广义相对论重构，因此可以利用为此类量子规范理论发展出的整套技术，特别是规范协变威尔逊圈变量 [30]。本文重点是概述学界为解决前述问题提出的不同研究方案，介绍各方案的优缺点与发展阶段。在“圈量子引力动力学综述”一节，我们首先会为过去三十年的各类发展提供一份总览路线图，我们尽可能做到表述非专业化。相关研究主要分为两大路径: 量子约束方案与约化相空间方案，分别在量子化后、量子化前对理论施加约束。这两种方案一般在半经典层面等价，但量子修正可能不同，因此该选择会影响相应的唯象学。在“量子化后施加约束”一节，我们详细介绍量子约束方案。在“量子化前施加约束”一节，我们详细介绍约化相空间方案。最后在“总结与展望”一节给出总结与结论。

Survey of LQG Dynamics

LQG 动力学研究综述

Over the past three decades, a substantial amount of knowledge has been collected of how to address the quantum dynamics of GR in the LQG approach. Progress has been possible because one insisted on background independence and non-perturbative techniques. For the first time, precise mathematical questions could be phrased and often answered in a concrete Hilbert space context. These developments are documented in several 100 publications with gradual improvements of each other over time resulting in different versions of the theory, making it difficult for the newcomer to evaluate the current status of the theory. The present section therefore is to serve as a chart.

过去三十年间，人们在 LQG 框架下研究广义相对论量子动力学，积累了大量成果。这些进展之所以能够取得，得益于对背景独立性和非微扰技术的坚持。我们首次得以在具体希尔伯特空间语境下提出精确的数学问题，且多数问题得到了解答。这些研究成果记载在数百篇文献中，理论版本随时间逐步改进，对入门者而言很难评估该理论的当前现状，因此本节作为内容总览。

1. Quantization before constraining This means that one first quantizes the full, unconstrained phase space and then imposes the constraints as operators in the corresponding representation of the canonical (or anti-)commutation relations (CCR or CAR) and adjointness relations (AR). In more details, this involves the following.

1. 约束前量子化该方法指首先对整个无约束相空间进行量子化，再在正则 (或反) 对易关系 (CCR 或 CAR) 与伴随关系 (AR) 的对应表示中将约束作为算符引入。具体包含以下步骤。

1.A Kinematical representations of CCR and AR

1.A CCR 与 AR 的运动学表示

In the gauge theory framework just mentioned, the unconstrained phase space is coordinatized by a pair (A, E) consisting of a real-valued SU(2) connection A and a real-valued non-Abelian electric field E which is the conjugate momentum of A . The set of constraints D, C is augmented by a Gauß constraint G . Then we look for a Hilbert space representation $(\rho, \mathcal{H})$ of the CCR, schematically $[A, A] = [E, E] = 0, [E, A] = i \cdot 1$, and the AR, schematically $A - A^* = E - E^* = 0$. To simplify the notation, here and in what follows, we do not distinguish between abstract algebra elements such as A and its operator representative $\rho(A)$. In quantum field theories (QFT), the number of unitarily inequivalent representations is infinite; hence, additional physical input, usually using details of the dynamics of the QFT under consideration, is needed to select suitable ones. In LQG one has good motivation to use the criterion of spatial diffeomorphism covariance to select an essentially unique representation $(\rho, \mathcal{H})$ [4, 5, 42, 76] about much will be said in Chap. 83, "Emergence of Riemannian Quantum Geometry" of this book.

在上述规范理论框架中，无约束相空间由一对 (A, E) 坐标化: 分别是实值 SU(2) 联络 A ，和作为 A 共轭动量的实值非阿贝尔电场 E 。约束集合 D, C 由高斯约束 G 补充。随后我们寻找 CCR(形式为 $[A, A] = [E, E] = 0, [E, A] = i \cdot 1$) 和 AR(形式为 $A - A^* = E - E^* = 0$) 的希尔伯特空间表示 $(\rho, \mathcal{H})$ 。为简化记号，在下文中我们不区分 A 这类抽象代数元素与其算符表示 $\rho(A)$ 。在量子场论 (QFT) 中，幺正不等价表示的数量是无穷多；因此需要额外的物理输入 (通常借助所研究量子场论的动力学细节) 来筛选合适的表示。在 LQG 中，我们有充分的动机利用空间微分同胚协变性准则选取一个本质唯一的表示 $(\rho, \mathcal{H})$ [4, 5, 42, 76] ，本书第 83 章“黎曼量子几何的涌现”会对此展开详细讨论。

1.B Induced representation of the constraint algebra

1.B 约束代数的诱导表示

The challenge is then to find operator representatives of the constraints G, D, C in that selected representation. Naively, one just needs to replace in the expression of the classical constraint function the variables A, E by the corresponding operators. However, this is non-trivial in several aspects. First, as the constraints are not linear functionals in A, E , an operator ordering of the constraints has to be chosen. Second, as A, E are not operators but operator-valued distributions on the spatial slice, we meet the problem to define the product of such distributions common to all QFT. Third, in quantum gravity, it is much worse than that because the constraints are not local polynomials but local algebraic functions of A, E and thus we need to define quotients, square roots, and even more singular algebraic functions of distributions. Fourth, all of this has to be done in such a way that the constraints represent $\mathfrak{h}$ without anomalies on a common, dense, invariant domain $\mathcal{D} \subset \mathcal{H}$ for all constraints.

当前的挑战是在选定的表示中找到约束 G, D, C 的算符对应形式。简单来说，我们只需要在经典约束函数的表达式中将变量 A, E 替换为对应的算符即可。但这在多个层面都并非易事: 首先，由于约束不是 A, E 中的线性泛函，必须为约束选择一种算符排序。其次，由于 A, E 不是算符，而是空间切片上的算符值分布，我们会遇到定义所有量子场论都普遍存在的分布乘积的问题。第三，在量子引力中，情况要糟糕得多: 约束不是局部多项式，而是 A, E 的局部代数函数，因此我们需要定义分布的商、平方根，甚至更奇异的代数函数。第四，所有这些操作都必须保证，约束在所有约束共有的稠密不变定义域 $\mathcal{D} \subset \mathcal{H}$ 上无反常地表示 $\mathfrak{h}$。

A solution to the first three challenges has been given for the first time in [92, 94] for both Euclidian and Lorentzian signature, with and without cosmological constant, with and without standard matter. This was later extended to any dimension and to supergravity [23-26], Chap. 85, ”Hamiltonian Theory: Generalizations to Higher Dimensions, Supersymmetry, and Modified Gravity”. More precisely, one first defines regulated constraints such as C_ε where ε is a short-distance cut-off, defines C_ε on the span of spin network functions (SNWF) $\mathcal{D}$ [85], and then takes the limit $\varepsilon \to 0$ in an operator topology exploiting spatial diffeomorphism covariance and invariance, thus arriving at continuum operators C densely defined on $\mathcal{D}$. The algebra of these quantum constraints is non-Abelian and in fact closes, but it closes with the wrong operator equivalents of the structure functions, although the commutator still annihilates spatially diffeomorphism-invariant states.

针对欧几里得号差和洛伦兹号差、带宇宙学常数和不带宇宙学常数、带标准物质和不带标准物质的情况，文献 [92, 94] 已经首次给出了前三项挑战的解决方案。后来该方案被推广到任意维度和超引力 [23-26]，参见第 85 章《哈密顿理论: 向高维、超对称和修正引力的推广》。更具体地说，研究者首先定义正则化约束，例如 C_ε ，其中 ε 是短距离截断，在自旋网函数 (SNWF) 张成的空间 $\mathcal{D}$ 上定义 C_ε [85]，随后利用空间微分同胚协变性和不变性在算符拓扑中取极限 $\varepsilon \to 0$ ，最终得到在 $\mathcal{D}$ 上稠密定义的连续统算符 C 。这些量子约束的代数是非阿贝尔的，并且实际上是闭合的，但闭合所用结构函数对应的算符等价形式是错误的，尽管对易子仍然零化空间微分同胚不变态。

To also accomplish the fourth challenge in [73, 122, 123] , it is proposed to represent $\mathfrak{h}$ not on $\mathcal{D}$ but instead on an invariant subspace (so-called habitat [44]) of $\mathcal{D}^*$, the space of algebraic (i.e., discontinuous) distributions on $\mathcal{D}$, at the price of changing the density weight of C . At the moment, this approach is geared to the Euclidian signature vacuum theory because one exploits the fact that in this case, C very much looks like D but with an electric field-dependent vector field as generator of one-parameter families of diffeo-morphisms (so-called electric shift approach). Another approach to the fourth challenge is (Hamiltonian) renormalization [99] which is complementary to the renormalization (see Chap. 93, ”Spin Foams, Refinement Limit, and Renormalization” and Ref. [90]) in the covariant (spin foam) approach to LQG. As pointed out in [100], the very definition of $\mathfrak{h}$ in the classical theory relies on the assumption that the metric q be regular. On the other hand, the quantum metric annihilates the dense domain $\mathcal{D}$ almost everywhere. The difficulties to accomplish the fourth challenge turn out to be tightly connected with the issue of quantum non-degeneracy, and it is therefore suggested to make quantum non-degeneracy an integral part of notion of anomaly freeness. By construction, the renormalization program systematically builds such nondegenerate representations of the CCR and AR. Yet another approach to the fourth challenge is the master constraint proposal; see immediately below.

为了在 [73, 122, 123] 中也完成第四项挑战，有研究提出不在 $\mathcal{D}$ 上表示 $\mathfrak{h}$，转而在 $\mathcal{D}^*$ 的不变子空间 (即所谓的环境空间 [44]) 上表示，该空间是 $\mathcal{D}$ 上代数 (即不连续) 分布的空间，代价是需要改变 C 的密度权。目前该方法适配欧几里得号差真空理论，因为它利用了该情形下 C 形式非常接近 D 的特点，仅差一个作为单参数微分同胚族生成元、依赖电场的矢量场 (即所谓的电移位方法)。应对第四项挑战的另一种方法是 (哈密顿) 重整化 [99]，它与 LQG 协变 (自旋泡沫) 方法中的重整化 (参见第 93 章《自旋泡沫、精化极限与重整化》和文献 [90]) 互补。正如文献 [100] 指出，经典理论中 $\mathfrak{h}$ 的定义本身就依赖于度规 q 正则的假设。另一方面，量子度规几乎处处零化稠密定义域 $\mathcal{D}$。事实上，完成第四项挑战遇到的困难与量子非退化问题紧密相关，因此有研究建议将量子非退化纳入无反常概念的核心部分。根据构造，重整化纲领会系统地构造 CCR 和 AR 的这种非退化表示。应对第四项挑战还有主约束方案，见下文。

1.C Kernel of the quantum constraints

1.C 量子约束的核

Assuming that all four challenges of the previous task have been met, the next step is to solve the constraints and to pass from the unphysical or kinematical Hilbert space $\mathcal{H}$ to the physical Hilbert space $\mathcal{H}_{\text{phys}}$. Unless zero is only in the pure point spectrum of all constraints, the physical Hilbert space is not a subspace of the kinematical one; thus, the formal solutions ψ to the constraint equations $G\psi = D\psi = C\psi = 0$ must be equipped with a new scalar product. If the constraints would form a Lie algebra of self-adjoint operators, then one could use rigging or group averaging techniques [60] which essentially is the map $\eta\psi = \delta(G, D, C)\,\psi$ where the δ distribution is obtained by passing from $\mathfrak{h}$ to the Lie group $\mathfrak{H}$ and integrating the unitary operators corresponding to the constraints over the Lie group. The new inner product is then essentially $<\eta\psi, \eta\psi'>_{\text{phys}} =<\psi, \eta\psi'>$.

假设前述任务的全部四个挑战都已解决，下一步就是求解约束，并从非物理/运动学希尔伯特空间 $\mathcal{H}$ 过渡到物理希尔伯特空间 $\mathcal{H}_{\text{phys}}$。除非零仅属于所有约束纯点谱的一部分，否则物理希尔伯特空间不是运动学希尔伯特空间的子空间；因此，约束方程 $G\psi = D\psi = C\psi = 0$ 的形式解 ψ 必须配备一个新的标量积。如果约束构成自伴算子的李代数，那么就可以使用装配或群平均技术 [60]，该技术本质上是映射 $\eta\psi = \delta(G, D, C)\,\psi$，其中通过从 $\mathfrak{h}$ 过渡到李群 $\mathfrak{H}$，并对约束对应的幺正算子在李群上积分得到 δ 分布，新内积本质上就是 $<\eta\psi, \eta\psi'>_{\text{phys}} =<\psi, \eta\psi'>$。

As this is not the case, one has two options. The first is the master constraint approach [109]; the second is to pass to classically to equivalent constraints (typically Abelian) that do form a Lie algebra [65].

由于实际情况并非如此，因此有两种可行方案: 第一种是主约束方案 [109]；第二种是转换为经典意义下等价的约束 (通常为阿贝尔约束)，这类约束确实可以构成李代数 [65]。

As far as the master constraint is concerned, the observation is that instead of considering the individual constraints, one can equivalently construct their weighted sum (or rather integral) M of squares called master constraint. The classical constraint surface of the phase space is then encoded by the single equation $\mathbf{M} = 0$, and a classical observable O is specified by the single equation $\{\{\mathbf{M}, O\}, O\}_{\mathbf{M}=0} = 0$. In the quantum theory, one then correspondingly also has to impose only a single equation $M\psi = 0$ which can be solved using rigging or direct integral decomposition (DID) [38] techniques. An attractive feature of this approach is that, since there is only one constraint left, one no longer has to worry about the representation of $\mathfrak{h}$ and its possible anomalies. However, while the spectrum of $\mathbf{M}$ is granted to be a subset of the non-negative reals, zero may not be included, and therefore, one may need to subtract from M a constant to allow for sufficiently many solutions. That constant is therefore a manifestation of the anomaly of $\mathfrak{h}$ in the master constraint approach.

就主约束而言，我们发现: 不考虑单个约束，也可以等价地构造它们平方的加权和 (更准确地说是积分) M ，即主约束。相空间的经典约束曲面由单个方程 $\mathbf{M} = 0$ 描述，经典可观测量 O 由单个方程 $\{\{\mathbf{M}, O\}, O\}_{\mathbf{M}=0} = 0$ 定义。在量子理论中，相应地只需要施加单个方程 $M\psi = 0$ ，可以通过装配技术或直和分解 (DID)[38] 求解。该方案的一个吸引人的特点是，由于仅剩下一个约束，因此无需再担心 $\mathfrak{h}$ 的表示及其可能的反常。然而，尽管 $\mathbf{M}$ 的谱必然属于非负实数的子集，但零可能不包含在其中，因此可能需要从 M 中减去一个常数以获得足够多的解。因此，该常数是主约束方案中 $\mathfrak{h}$ 反常的体现。

As far as constraint Abelianization is concerned, the idea is to solve the constraints, collectively called Z , for certain momenta p , i.e., to write them in the equivalent form $\widetilde{Z} = p + h$ where h does not depend on p . Then one can show that the $\widetilde{Z}$ have vanishing Poisson brackets among themselves [66] and therefore rigging techniques can be used in principle if one is able to find a representation of that Abelian algebra.

就约束阿贝尔化而言，其核心思想是对统称为 Z 的约束求解部分动量 p ，也就是将约束改写为等价形式 $\widetilde{Z} = p + h$ ，其中 h 不依赖于 p 。此时可以证明 $\widetilde{Z}$ 相互之间的泊松括号为零 [66]，因此只要能找到该阿贝尔代数的一个表示，原则上就可以使用装配技术。

1.D Induced representation of the quantum observables

1.D 量子可观测量的诱导表示

Finally, supposing that the physical Hilbert space has been constructed, we still have not done any physics with it. Abstract quantum observables are the self-adjoint operators O defined densely on $\mathcal{H}_{\text{phys}}$. Thus, they preserve the joint kernel of the constraints and thus have vanishing commutators with all constraints on that kernel. However, a priori, they lack any physical interpretation, and thus, useful observables will be representations of classical observables (vanishing Poisson brackets with all constraints when the constraints hold). This will again meet the same mathematical challenges as the constraints themselves because the observables of GR are expected to be non-polynomial (and in case of vacuum GR even non-local [117]). Also, one has to extract a non-trivial physical time evolution (or physical Hamiltonian) for those observables which reduces to the time evolution of (standard) matter observables when fluctuations of GR around exact vacuum solutions (e.g., Minkowski) are negligible.

最后，假设我们已经构造出物理希尔伯特空间，仍尚未用它开展任何物理研究。抽象量子可观测量是稠密定义在 $\mathcal{H}_{\text{phys}}$ 上的自伴算子 O 。因此它们保持约束的联合核，在该核上与所有约束的对易子为零。但它们先天缺乏物理解释，因此有用的可观测量是经典可观测量的表示 (当约束成立时，其与所有约束的泊松括号为零)。这会再次遇到和约束本身相同的数学难题，因为广义相对论的可观测量一般都是非多项式的 (真空广义相对论中甚至是非局域的 [117])。此外，我们还需要为这些可观测量提取非平凡的物理时间演化 (或物理哈密顿量)，使得当广义相对论在精确真空解 (例如闵氏时空) 周围的涨落可忽略时，该演化退化为 (标准) 物质可观测量的时间演化。

2. Quantization after constraining

2. 约束后量子化

In the reduced phase space approach, one aims at quantizing the physical degrees of freedom only. For this purpose, one solves the constraints already at the classical level by means of constructing classical observables and takes their corresponding algebra as the starting point for the quantization of the CCR and AR. The dynamics of these observables is generated by a so-called physical Hamiltonian that will be implemented as a corresponding operator directly on the physical Hilbert space. Therefore, non-observable quantities are never subject to quantization, and anomalies cannot arise. Given the severe difficulties of the quantization before constraining approach sketched above, the reduced phase space approach appears to be especially attractive and much more economic. However, whether the quantization program can be completed along these lines crucially depends on finding a representation of the CCR and AR of these observables because in general, these algebras have a more complicated structure than their kinematical counterparts of the non-observables A, E . Those observables can be explicitly constructed in the framework of the relational formalism [36, 83, 110, 124, 125]. A key ingredient is to choose some reference fields as ”clocks” and ”rods” as a physical dynamical reference system with respect to which the observables and dynamics of the remaining degrees of freedom are formulated. What kind of reference fields are chosen here is free at the first place and determines the reduced model as the reduced phase space and the physical Hamiltonian depend on these choices. This is of course expected; even on Minkowski space, the Hamiltonians of a free Klein-Gordon field depend on the inertial frame that one uses, differing by a corresponding boost generator. Often, one distinguishes between so-called geometrical and matter reference fields that are chosen from the corresponding sectors of the theory, but of course, a mixture will also be possible. In full vacuum GR, such a manageable *-algebra has not been found to date. So far, those geometrical clocks have only been considered on a perturbative level [37, 43, 58, 69] which is why most efforts in that direction use matter in a crucial way.

在约化相空间方法中，目标是仅对物理自由度进行量子化。为此，人们已经在经典层面通过构造经典可观测量求解约束，并将其对应的代数作为对 CCR 和 AR 进行量子化的起点。这些可观测量的动力学由所谓的物理哈密顿量生成，该哈密顿量将作为对应的算符直接作用在物理希尔伯特空间上。因此，非可观测量永远不需要经过量子化步骤，也不会产生反常。鉴于上文概述的约束前量子化方法存在重重困难，约化相空间方法显得格外有吸引力，也更加简洁高效。然而，量子化方案能否沿这一思路完成，关键取决于能否找到这些可观测量的 CCR 和 AR 表示，因为一般而言，这类代数的结构比非可观测量对应的运动学代数更复杂 A, E 。这些可观测量可以在关系形式论的框架下显式构造 [36, 83, 110, 124, 125]。该方法的一个核心要点是选取参考场作为“时钟”和“标尺”，构成一个物理动力学参考系，剩余自由度的可观测量和动力学都是相对于该参考系定义的。这里选取何种参考场最初并没有限制，而约化模型的性质由该选择决定——因为约化相空间和物理哈密顿量都依赖于这些选择。这一点当然符合预期: 即使在闵可夫斯基空间中，自由克莱因-戈登场的哈密顿量也依赖于所使用的惯性系，不同惯性系下的哈密顿量相差一个对应的 boost 生成元。人们通常会区分从理论对应 sector 中选取的所谓几何参考场和物质参考场，但当然也可以混合选取。在完整的真空广义相对论中，迄今尚未找到这样易于处理的 *-代数。到目前为止，这类几何时钟仅在微扰层面得到研究 [37, 43, 58, 69] ，这也是该方向的大多数研究都将物质作为核心要素的原因。

2.A. Choice and phenomenology of material reference systems

2.A. 物质参考系的选择与唯象学

Working with physical dynamical reference systems takes into account that in realistic models, something like an idealized test observer that does not back react on the system does not exist and can at most only be assumed in some limit of the models. In order to serve as a system of ”rods and clocks,” the matter must be present ”everywhere and every time.” In other words, it must have non-vanishing energy momentum density throughout spacetime, and considered as a map from M to itself, it should be a diffeomorphism. Given a system to start that couples matter dynamically to gravity, one chooses in general a suitable subset of the matter degrees as reference fields and then constructs the reduced theory of the remaining physical observables. Choosing a different subset as matter reference fields yields to another reduced theory with the same amount of physical degrees of freedom, and at the classical level, the relation among these two reduced models is well established. Not any chosen matter type will necessarily yield to a manageable observable algebra for which the quantization program can be completed, and therefore, it is often convenient to choose a reference matter model for which we can mimic in some limit the idealized test observer because it influences the ”observed matter and geometry ” as little as possible. A matter species that appears to come close to have those properties is dark matter (DM) [3] for it interacts only gravitationally and seems to be omnipresent throughout the universe, although presently it is unclear what DM actually is. In the LQG literature, several matter types were employed, mostly driven by mathematical convenience, and these matter types were coupled in addition to standard model matter. Typically, these are scalar fields with an energy momentum tensor of perfect fluid form, with dust matter being not only particularly convenient but also because it comes closest to the notion of a test observer that only interacts gravitationally [27,72]. For instance, in the case of cosmology, this means one works with two-fluid models instead of one-fluid models [59]. Since by construction such models carry more physical degrees of freedom than the corresponding models without the reference matter, one needs to carefully analyze the phenomenological implication of these reduced models, thereby possibly constraining the (dark) matter model employed. This has been done, e.g., in [53,54] where it is shown that for the Einstein-inflaton system with four additional dust fields, the additional polarizations here assigned to the gravitational sector rapidly decay in the late universe. Furthermore, in [59], the effect of different choices of clocks on inflation was investigated showing that there exist initial conditions for which dust and scalar field clocks behave as ”good” clocks that influence the observed system only in a tiny way.

使用物理动力学参考系需要考虑到: 在现实模型中，不存在对系统不产生反作用的理想化测试观测者，这类观测者最多只能在模型的某些极限中假设存在。要成为一套“尺与钟”系统，物质必须“无处不在、无时不有”。换句话说，物质在整个时空的能量动量密度都不为零，且将其视为从 M 到自身的映射时，它应当是一个微分同胚。对于一个初始包含物质与引力动力学耦合的系统，通常我们会选择合适的物质自由度子集作为参考场，随后构造剩余物理可观测量的约化理论。选择不同的物质参考场子集会得到另一个约化理论，其物理自由度的数量不变，且在经典层面，这两个约化模型之间的关系已经得到明确。并非任意选定的物质类型都必然能得到可处理、可完成量子化方案的可观测量代数，因此，选择一种能在某些极限下模拟理想化测试观测者的参考物质模型通常更为方便，因为这类模型对“被观测物质与几何”的影响尽可能小。暗物质 (DM)[3] 似乎是一类接近满足这些性质的物质种类: 它仅通过引力相互作用，且似乎遍布整个宇宙，尽管目前我们尚不清楚暗物质的本质究竟是什么。在圈量子引力 (LQG) 文献中，已经使用过多种物质类型，这类选择大多出于数学便利，且这些物质类型是附加在标准模型物质之外耦合的。典型的参考物质是具有完美流体形式能量动量张量的标量场，其中尘质物质不仅特别方便，而且最接近仅发生引力相互作用的测试观测者概念 [27,72]。例如在宇宙学情形中，这意味着我们使用双流体模型而非单流体模型 [59]。由于通过构造可知，这类模型比不包含参考物质的对应模型拥有更多物理自由度，因此我们需要仔细分析这些约化模型的唯象学推论，这可能会对所使用的 (暗) 物质模型产生约束。这类分析已有先例，例如文献 [53,54] 表明，对于引入四个额外尘场的爱因斯坦-暴胀子系统，分配给引力部分的额外偏振在宇宙晚期会迅速衰减。此外，文献 [59] 研究了不同时钟选择对暴胀的影响，结果表明存在满足条件的初始条件，使得尘时钟和标量场时钟表现为“好”时钟，仅对被观测系统产生极小的影响。

2.B Deriving the reduced phase space

2.B 推导约化相空间

Once suitable reference matter, the reduced phase space is obtained by rewriting the set of constraints in an equivalent form that is convenient for the construction of the observables.

选定合适的参考物质后，我们将约束集改写为等价形式，即可得到约化相空间，这种形式便于构造可观测量。

Abelianization of the constraints:

约束的阿贝尔化:

For each constraint, one reference matter field is chosen, and this field needs to be at least weakly canonically conjugate to the corresponding constraint. In the matter models considered in [27, 39, 50, 55, 57, 68, 72] , this can be obtained by solving the constraints, denoted by Z_I , for the reference matter momenta π_I conjugate to the material reference matter ϕ^I where I labels the set of constraints. A physically equivalent form of the constraints is then given by $\widetilde{Z}_I = \pi_I + h_I$ where h_I depends on all variables of the unconstrained phase space except the π_I . This set of constraints is then by construction Abelian using that the original constraints are first class. While in principle such a (weak) Abelianization is always possible, at least locally, it usually requires to solve PDEs to obtain the $\widetilde{Z}_I$ if the momenta π_I are not involved only algebraically in the Z_I . The latter is the case for the gravitational degrees of freedom and explains why using matter is practically important. As the standard dependence in the Hamiltonian constraint on π_I is typically quadratic, a drawback is that the phase space decomposes into branches related to the sign ambiguities of h . As a consequence, $Z_I, \widetilde{Z}_I$

are strictly equivalent only on one of those branches. This may be considered a mild price to pay in view of the huge amount of advantages that one otherwise gains over the quantization before constraining method.

对每个约束选择一个参考物质场，该场至少需要与对应约束弱正则共轭。在 [27, 39, 50, 55, 57, 68, 72] 研究的物质模型中，可通过求解约束 Z_I，得到参考物质动量 π_I，π_I 共轭于物质参考系物质 ϕ^I，其中 I 标记约束集合。由此得到物理上等价的约束形式为 $\widetilde{Z}_I = \pi_I + h_I$，其中 h_I 依赖于未约束相空间中除 π_I 外的所有变量。利用原约束为第一类约束的性质，该约束集合通过构造就是阿贝尔的。原则上这种 (弱) 阿贝尔化始终可行 (至少局部可行)，但若 π_I 并非仅代数出现在 Z_I 中，通常需要求解偏微分方程才能得到 $\widetilde{Z}_I$。引力自由度就属于这种情况，这也解释了使用物质为何在实际中十分重要。由于哈密顿约束中对 π_I 的标准依赖通常是二次的，一个缺点是相空间会因 h 的符号歧义分解为多个分支。因此，$Z_I, \widetilde{Z}_I$ 仅在其中一个分支上严格等价。考虑到该方法相较于约束前 quantization 能带来大量优势，这可被视为一个微不足道的代价。

Construction of observables and physical Hamiltonians:

可观测量与物理哈密顿量的构造:

To construct observables, one introduces a set of gauge fixing conditions, denoted by $F_I = \phi^I - \tau^I$ where τ^I are fixed functions on the spacetime manifold M sometimes called "multi-fingered" times. The constraints written in the form $\widetilde{Z}_I$ have the property that their corresponding "Dirac matrix" $M^I \ := \{F^J, \widetilde{Z}_J\}$ has a non-vanishing determinant being the requirement admissible gauge fixing conditions need to satisfy. Denoting the remaining conjugate pair, sometimes called "true degrees of freedom," collectively by (Q, P), suppressing any index structure here, one can construct a "relational" observable [36, 83, 110, 124, 125] O_K associated with a generic phase space functional K of (Q, P) as $O_K(\tau) \ := \left[\ \exp\left(uX_{\widetilde{Z}}\right) \ \cdot \ K \ \right]\Big|_{u=F}$ with $X_{\widetilde{Z}}$ being the Hamiltonian vector field of the constraint $\widetilde{Z}$. Its interpretation is that it returns the value of K when ϕ takes the value τ underlying their relational nature. In particular, O_K agrees with K in the gauge $F = 0$ and thus can be understood as the corresponding gauge-invariant extension of K. The so obtained O_K s have vanishing Poisson brackets with all constraints $\widetilde{Z}$ and are thus gauge invariant. Since K, K' depend only on Q, P, we have $\{O_K, O_{K'}\} = O_{\{K,K'\}}$ for the equal τ brackets, i.e., the CCR and AR of K, K' and $O_K, O_{K'}$, respectively, are isomorphic. Consequently, if one considers Q, P, one can conclude that O_Q, O_P remain canonically conjugate and thus retain simple CCR and AR meaning that quantizing the observable algebra is not more difficult than the corresponding kinematical algebra. The dynamics of the O_K can be understood as a physical evolution with respect to the parameter τ_0 involved in gauge fixing condition $F_0 = \phi_0 - \tau_0$ associated with the Hamiltonian constraint $C = \pi_0 + h_0$, where the label zero was introduced to label the corresponding quantities for the Hamiltonian constraint. The generator of the dynamics, the physical Hamiltonian, is given by the observable $H \ := \ O_{h_0}$ that is a functional of the observables O_Q, O_P only and can in principle depend on τ_0 and thus be a time-dependent Hamiltonian. Its explicit form depends on the chosen reduced model, and specific examples will be discussed in section "Constraining Before Quantization".

为了构造可观测量, 需要引入一组规范固定条件, 记为 $F_I = \phi^I - \tau^I$, 其中 τ^I 是时空流形 M 上的固定函数, 这类函数有时被称为“多指”时间。写成 $\tilde{Z}_I$ 形式的约束具有如下性质: 其对应的“狄拉克矩阵” $M^I := \{F^J, \tilde{Z}_J\}$ 具有非零行列式, 这是可容许规范固定条件需要满足的要求。将剩余的共轭对 (有时称为“真实自由度”) 整体记为 (Q,P) ,此处省略所有指标结构,我们可以为 (Q,P) 的一般相空间泛函 K 构造关联的“关系”可观测量 [36, 83, 110, 124, 125] O_K , 形式为 $O_K(\tau) := \left[\ \exp\left(uX_{\tilde{Z}}\right) \ \cdot\ K\ \right]\Big|_{u=F}$ ，其中 $X_{\tilde{Z}}$ 是约束 $\tilde{Z}$ 的哈密顿矢量场。它的解释是: 当 ϕ 取值为 τ 时，它返回 K 的值，这是其关系性质的基础。特别地，O_K 在规范 $F = 0$ 下与 K 一致，因此可以理解为 K 对应的规范不变延拓。这样得到的所有 O_K 与所有约束 $\tilde{Z}$ 的泊松括号均为零，因此是规范不变的。由于 K, K' 仅依赖于 Q, P ，对相等 τ 括号我们有 $\{O_K, O_{K'}\} = O_{\{K,K'\}}$ ，即 K, K' 和 $O_K, O_{K'}$ 分别对应的正则对易关系 (CCR) 和代数关系 (AR) 是同构的。因此，如果考虑 Q, P ，可以得出结论: O_Q, O_P 保持正则共轭，因此保留了简单的 CCR 和 AR，这意味着可观测量代数的量子化并不比相应的运动学代数更困难。O_K 的动力学可以理解为相对于参数 τ_0 的物理演化，τ_0 出现在与哈密顿约束 $C = \pi_0 + h_0$ 关联的规范固定条件 $F_0 = \phi_0 - \tau_0$ 中，其中零标号是用来标记哈密顿约束对应量的。动力学的生成元即物理哈密顿量由可观测量 $H := O_{h_0}$ 给出，$H := O_{h_0}$ 仅是可观测量 O_Q, O_P 的泛函，原则上可以依赖于 τ_0 ，因此是含时哈密顿量。其具体形式依赖于所选的约化模型，具体例子将在“量子化前约束”一节中讨论。

2.C Quantization of the reduced phase

2.C 约化相的量子化

The final step consists in picking a representation $(\rho_{\text{phys}}, \mathcal{H}_{\text{phys}})$ of the CCR and AR of (Q, P) . Again, this needs physical input. As we are now in the situation of an ordinary Hamiltonian system, it is natural to impose the condition that H when expressed in terms of the operators corresponding to Q, P is promoted to a densely defined self-adjoint operator on $\mathcal{H}_{\text{phys}}$ after suitable regularization and renormalization steps. Since one can choose (Q, P) to be (A, E) , one can pick the same representation as in the quantization before constraining route just that now the Hilbert space is the physical Hilbert space and not the kinematical one.

最后一步是选取 $(\rho_{\text{phys}}, \mathcal{H}_{\text{phys}})$ 对 (Q,P) 的正则对易关系与约束代数的一个表示。这一步同样需要物理输入。由于我们现在处理的是普通哈密顿系统，很自然会要求: 当 H 用对应于 Q, P 的算符表示时，经过合适的正则化和重整化步骤后，它会被提升为 $\mathcal{H}_{\text{phys}}$ 上一个稠定自伴算符。由于我们可以将 (Q,P) 选为 (A,E) ，因此可以采用约束前量子化路线中相同的表示，区别仅在于此处的希尔伯特空间是物理希尔伯特空间，而非运动学希尔伯特空间。

Defining the Hamiltonian in that representation meets the same mathematical challenges as for defining the constraints themselves, depending on the matter employed, these are even worse because of the additional square roots involved in the expression for h which is an algebraic function of q, G, D, C . Nevertheless, in [39,50,55,57,68], it was shown that using the same methods as in [92,94], one can arrive at a positive operator H on $\mathcal{H}_{\text{phys}}$ which therefore has at least one self-adjoint (Friedrichs) extension. Moreover, the operator H , just as its classical analog, is Gauß and spatially diffeomorphism invariant (with respect to active rather than passive diffeomorphisms) and therefore free of part of the possible anomalies.

在该表示中定义哈密顿量，会遇到与定义约束本身相同的数学问题；根据所采用的物质内容不同，这些问题甚至会更严重，因为作为 q, G, D, C 代数函数的 h 的表达式中包含额外的平方根。尽管如此，文献 [39,50,55,57,68] 表明，采用与 [92,94] 相同的方法，可以在 $\mathcal{H}_{\text{phys}}$ 上得到正算符 H，因此它至少存在一个自伴 (弗里德里希) 扩张。此外，H 和它的经典类比一样，满足高斯不变性和空间微分同胚不变性 (针对主动微分同胚而非被动微分同胚)，因此不存在部分可能的反常。

A crucial difference between the action of H and the constraints is the following: The Hilbert space is a direct sum of mutually orthogonal subspaces which are labeled by piecewise analytic (more precisely semi-analytic; see Chap. 83, ”Emergence of Riemannian Quantum Geometry”) graphs. Thus, the Hilbert space is not separable. Due to this, the fact that the (active) spatial diffeomorphisms mix those sectors, and since H is spatially diffeomorphism invariant, the operator H must not mix those sectors; otherwise, it could not be densely defined. This feature has been coined ”non-graph changing” and applies to any spatially diffeomorphism-invariant operator. By contrast, the constraint operators are not spatially diffeomorphism invariant but just covariant; thus, their action does mix those sectors and are thus called ”graph changing.” This difference has important consequences for the semiclassical limit with respect to coherent states [103, 104, 107, 112] on fixed graphs: While the semiclassical limit of H coincides with the classical expression plus quantum corrections for sufficiently fine and large graphs, the semiclassical limit of the constraints with respect to the same states does not yield the expected result. Presently, no semiclassical states are known with respect to which the constraints have an appropriate limit, but superpositions of the coherent states over those graphs generated by the constraints appear to be promising candidates.

H 和约束算符的作用存在一个关键区别: 希尔伯特空间是相互正交子空间的直和，这些子空间由分段解析 (更准确说是半解析；参见第 83 章“黎曼量子几何的涌现”) 图标记。因此，该希尔伯特空间是不可分的。由此，(主动) 空间微分同胚会混合这些扇区，而由于 H 是空间微分同胚不变的，因此 H 算符不能混合这些扇区；否则它无法成为稠定算符。这个特性被称为“不改变图”，所有空间微分同胚不变算符都满足该性质。与之相对，约束算符不满足空间微分同胚不变性，仅满足协变性；因此它们的作用会混合这些扇区，因此被称为“改变图”。这种区别对固定图上相干态的半经典极限有重要影响 [103, 104, 107, 112]: 对于足够精细、足够大的图，H 的半经典极限等于经典表达式加量子修正，而约束相对于相同态的半经典极限得不到预期结果。目前尚未找到能让约束得到合适极限的半经典态，但由约束生成的这些图上相干态的叠加看起来是很有希望的候选者。

The fact that H must be non-graph changing and appears to have good semiclassical behavior on sufficiently large and fine graphs has motivated the algebraic quantum gravity (AQG) version of LQG [48, 49, 51, 52] . Here, one considers a single, infinite abstract (algebraic) graph once and for all. The term ”algebraic” means that the graph is completely specified by its adjacency matrix encoding just the information which vertices are connected by which edges. All quantum degrees of freedom and the Hilbert space refer to those abstract edges and vertices only, thus defining an abstract *-algebra of operators. Also, the Hamiltonian H is expressed in terms of these algebra elements, and it now acts on the chosen abstract graph which it preserves by construction and thus is non-graph changing. However, it is not required to be sub-graph preserving and in fact does not. The information of how the abstract graph is embedded into the spatial hypersurface (knotting and braiding of edges) is supplied by the semiclassical state with respect to which the semiclassical limit is under good control. If the abstract graph is infinite, one can choose that embedding arbitrarily densely so that in this sense, a continuum limit is partially obtained.

H 必须满足不改变图性质，且在足够大、足够精细的图上具有良好的半经典行为，这一事实推动了圈量子引力 (LQG) [48, 49, 51, 52] 的代数量子引力 (AQG) 版本的发展。在代数量子引力框架中，我们只需要预先考虑一个单一的无穷维抽象 (代数) 图。“代数”一词意味着该图完全由其邻接矩阵刻画，邻接矩阵仅记录哪些顶点由哪些边连接的信息。所有量子自由度和希尔伯特空间都仅对应这些抽象边和顶点，由此定义了一个抽象的算子 * 代数。哈密顿量 H 也可以用这些代数元素表示，它作用在选定的抽象图上，按构造就能保持该图不变，因此满足不改变图性质。不过它不要求保持子图，实际上也确实不保持。抽象图如何嵌入空间超曲面 (边的扭结和编织) 的信息由半经典态提供，相对于该半经典态，半经典极限得到了良好控制。如果抽象图是无穷的，我们可以选择任意稠密的嵌入，因此从这个意义上说，我们已经部分得到了连续极限。

One additional advantage of AQG over LQG is that it reduces the degree of non-separability of the physical Hilbert space: In the case of compact spatial slices, it becomes separable; in the non-compact case, it is still nonseparable but in a more controlled fashion involving the so-called infinite tensor product (ITP) [105]. The non-separability of the (physical) Hilbert space is anyway a source of several complications such as the discontinuous action of spatial diffeomorphisms and holonomy operators and the fact that no SNWF functions exist which are excited everywhere on the spatial slice and which therefore describe a degenerate quantum geometry. It is also a vast over-coordinatization of the degrees of freedom as, e.g., much fewer graphs would suffice to separate the points in the space of classical connections. AQG therefore can be considered as the starting point for the Hamiltonian renormalization approach to LQG [99] where in addition one tries to remove the dependence on the abstract graph of AQG, thereby fully reaching the continuum limit.

AQG 相较于 LQG 的另一项优势在于，它降低了物理希尔伯特空间的不可分性程度: 在紧致空间切片的情况下，希尔伯特空间是可分的；在非紧致情况下，它仍然不可分，但借助所谓的无穷张量积 (ITP)[105] 变得更易于控制。物理希尔伯特空间的不可分性本就是诸多问题的根源，例如空间微分同胚和完整算符的作用不连续，也不存在能在整个空间切片上激发、从而描述简并量子几何的 SNWF 函数。同时它也对自由度造成了过度参数化，例如，少得多的图就足以分离经典联络空间中的点。因此 AQG 可被视为 LQG 哈密顿重整化方法的出发点 [99]，该方法额外尝试消除对 AQG 抽象图的依赖，从而完全达到连续统极限。

Another advantage of AQG over LQG is the issue of propagation [89]: In order that the commutator of Hamiltonian constraints annihilates spatially diffeomorphism-invariant distributions, the constraints defined in [92,94,95] are such that a second action does not react to the changes made to the graph underlying a SNWF that were done by the first action. This appears to induce an unphysical ultra-locality and thus absence of propagation in the sense that the Hamiltonian constraint action can be described in terms of its action at individual vertices that do not influence each other. That this is not the case beyond reasonable doubt has been demonstrated in [115] where it is pointed out that communication between vertices does happen at the level of the solutions to all constraints. The basic mechanism is that a solution is also spatially diffeomorphism invariant and therefore the constraint information coming from its action at adjacent vertices cannot be uniquely assigned to one or the other of those vertices any more, i.e., the information spreads out. In AQG, this propagation issue does not arise from the outset because the Hamiltonian is not sub-graph preserving.

AQG 相较于 LQG 的另一项优势体现在传播问题 [89] 上: 为了让哈密顿约束的对易子零化空间微分同胚不变分布，文献 [92,94,95] 中定义的约束具备这样的性质: 第二次作用不会响应第一次作用给 SNWF 基础图带来的改变。这似乎会引入非物理的超局域性，进而导致传播缺失——也就是说哈密顿约束作用可以通过它在互不影响的独立顶点上的作用来描述。文献 [115] 已经明确证实事实并非如此，该文指出，顶点间的相互作用在所有约束的解层面确实存在。基本机制是，解本身也是空间微分同胚不变的，因此来自相邻顶点作用的约束信息无法再唯一分配给其中任意一个顶点，换句话说，信息会扩散开。在 AQG 中，这种传播问题从一开始就不会出现，因为哈密顿不保持子图。

3. Connection with covariant quantization - path integrals

3. 与协变量子化的联系——路径积分

The precise correspondence between canonical and covariant quantization of constrained systems, no matter how complicated the constraints and their algebra may be, is by now well understood and subject of many excellent textbooks such as [66]. Basically, it starts from the reduced phase space description above which is an ordinary Hamiltonian system. Its phase space path integral description of inner products between physical states Ψ, Ψ' can therefore be obtained by standard methods of (Euclidian) QFT. Then one unfolds the reduced phase space path integral into a path integral over the full kinematical phase space using appropriate δ distributions and measure factors (essentially the square root of the modulus of the determinant of the Poisson bracket matrix between second-class constraints and the modulus of the determinant of the Poisson bracket matrix between the first-class constraints and the gauge fixing conditions). The application of this general framework to GR in the gauge theory description has been described in [65]. One can view the resulting formula as an explicit implementation of the rigging map $\Psi = \eta\psi, \Psi' = \eta\psi'$ for purely first-class constrained systems mentioned above with $\eta = \delta\left[\widetilde{Z}\right]$.

无论约束及其代数多么复杂, 约束系统的正则量子化与协变量子化之间的精确对应目前已得到充分理解，也是 [66] 等诸多优秀教材的讨论主题。该框架基本以上文介绍的约化相空间描述为起点，约化相空间描述是一个常规哈密顿系统。因此它对物理态之间内积的相空间路径积分描述 Ψ, Ψ' 可以通过 (欧氏) 量子场论的标准方法得到。之后我们利用合适的 δ 分布与测度因子 (本质上是第二类约束泊松括号矩阵行列式的模，以及第一类约束与规范固定条件泊松括号矩阵行列式的模，二者乘积的平方根)，将约化相空间路径积分展开为全运动学相空间上的路径积分。这个通用框架应用于规范描述下广义相对论的工作已在 [65] 中完成。我们可以将得到的公式视为前文提到的纯第一类约束系统 (对应 $\eta = \delta\left[\widetilde{Z}\right]$) 中 rigging 映射 $\Psi = \eta\psi, \Psi' = \eta\psi'$ 的显式实现。

It is important to stress that here we must use the Abelianized version $\widetilde{Z}$ of the constraints $Z = (G, D, C)$ as otherwise group averaging cannot be performed due to the fact that the algebra of the Z is not a Lie algebra. Note that this can be done for arbitrary (standard) matter coupling. The final path integral is over the unconstrained phase space. Only in simple cases can one carry out the momentum integrals and arrive at a path integral just over the kinematical configuration space.

需要强调的是，这里我们必须使用约束 $Z = (G, D, C)$ 的阿贝尔化版本 $\widetilde{Z}$ ，否则无法进行群平均，因为 Z 的代数不是李代数。请注意该操作适用于任意 (标准) 物质耦合。最终路径积分是在无约束相空间上进行的。只有在简单情况下才能完成动量积分，得到仅在运动学构型空间上的路径积分。

In the spin foam approach to LQG described in Chaps. 86, ”Spin Foams: Foundations” and -87, ”Spinfoams and High-Performance Computing" of this book, one chooses a different starting point. Namely, one postulates the rigging inner product as the path integral over A, B, C of the exponential of i times the Plebanski action and the boundary states ψ, ψ' which depend only on the Lorentz connection A (for the chosen signature). The Plebanski action contains a $B \wedge F$ term where F is the curvature of A and B is a Lie algebra-valued two-form and a term of the form $C \cdot B \wedge B$ where C is a Lagrange multiplier imposing certain simplicity constraints on B ensuring that B is derived from a tetrad (so that the BF terms becomes the Palatini action). To date, establishing the precise relation between the Plebanski version and the reduced phase derivation of the covariant formulation sketched above has not been achieved.

在本书第 86 章“自旋泡沫: 基础”与第 87 章“自旋泡沫与高性能计算”所描述的 LQG 自旋泡沫方法中，人们选择了不同的出发点: 即将 A, B, C 路径积分上的装配内积公设为 i 倍普莱班斯基作用量的指数，以及边界态 ψ, ψ'；边界态 ψ, ψ' 仅依赖于 (对应所选符号差的) 洛伦兹联络 A 。普莱班斯基作用量包含一项 $B \wedge F$ ，其中 F 是 A 的曲率，B 是李代数值二形式，还包含形如 $C \cdot B \wedge B$ 的项，其中 C 是拉格朗日乘子，它对 B 施加特定的简单性约束，确保 B 由标架导出 (因此 BF 项变为帕拉蒂尼作用量)。截至目前，我们仍未建立普莱班斯基版本与上文概述协变表述的约化相位推导之间的精确关系。

Constraining After Quantization

量子化后的约束条件

This section goes into more details with respect to the Dirac quantization approach of LQG. The reader is encouraged to study the original literature that we refer to along the presentation. We will consider only four spacetime dimensions and standard matter; see [23-26] and - Chap. 85, "Hamiltonian Theory: Generalizations to Higher Dimensions, Supersymmetry, and Modified Gravity" for how to pass beyond those limitations.

本节详细介绍圈量子引力 (LQG) 的狄拉克量子化方案。我们建议读者在阅读本文介绍的同时研究我们引用的原始文献。本文仅讨论四维时空和标准物质；若想了解如何突破这些限制，可参考 [23-26] 以及第 85 章“哈密顿理论: 向高维、超对称与修正引力的推广”。

Notation

记号

We begin with the notation:

我们首先介绍记号:

$a, b, c, .. = 1, .., 3$: Spatial tensor indices

$a, b, c, .. = 1, .., 3$: 空间张量指标

$A, B, C, .. = 1, 2$: Chiral fermion (spinorial) index

$A, B, C, .. = 1, 2$: 手征费米子 (旋量) 指标

$\varepsilon_{AB}, \varepsilon^{AB}$: Completely skew spinor metric

$\varepsilon_{AB}, \varepsilon^{AB}$: 全反对称旋量度规

$j, k, l, .. = 1, .., 3$ ： su(2) Lie algebra index

$j, k, l, .. = 1, .., 3$ ： su(2) 李代数指标

$J, K, L, .. = 1, .., d$: Lie algebra index of compact gauge group G of dimension d

$J, K, L, .. = 1, .., d$: 维数为 d 的紧致规范群 G 的李代数指标

$\iota, \kappa, \lambda = 1, .., N$: Defining representation index of compact gauge group G

$\iota, \kappa, \lambda = 1, .., N$: 紧致规范群 G 的基础表示指标

$\alpha, \beta, \gamma = 0, .., n-1$: Dark matter or dust species label

$\alpha, \beta, \gamma = 0, .., n-1$: 暗物质或尘埃种类标记

δ_{jk}, δ^{jk} : Kronecker symbol, similar $\delta_{JK}, \delta_{\iota\kappa}, \delta_{\alpha\beta}, \delta^{\alpha\beta}$

δ_{jk}, δ^{jk} : 克罗内克符号，类似 $\delta_{JK}, \delta_{\iota\kappa}, \delta_{\alpha\beta}, \delta^{\alpha\beta}$

A_a^j : Gravitational connection

A_a^j : 引力联络

E_j^a : Gravitational electric field; momentum conjugate to A

E_j^a : 引力电场；A 的共轭动量

Γ_a^j : Spin connection determined by E

Γ_a^j : 由 E 确定的自旋联络

Γ_{bc}^a : Levi-Civita connection determined by E

Γ_{bc}^a : 由 E 确定的列维-奇维塔联络

$\underline{A}_a^J$: Yang-Mills connection

$\underline{A}_a^J$: 杨-米尔斯联络

$\underline{E}^a_J$: Yang-Mills electric field; momentum conjugate to $\underline{A}$

$\underline{E}^a_J$: 杨-米尔斯电场；$\underline{A}$ 的共轭动量

η^A_ι : Chiral fermion field with YM charge

η^A_ι : 带杨-米尔斯电荷的手征费米子场

υ^A : Chiral fermion field without YM charge (neutrino singlet)

υ^A : 不带杨-米尔斯电荷的手征费米子场 (中微子单态)

ϕ_ι : Higgs field for G (generically complex valued)

ϕ_ι : G 对应的希格斯场 (一般为复数值)

π^ι : Conjugate momentum of Higgs field

π^ι : 希格斯场的共轭动量

X^α : Dark matter or dust scalars

X^α : 暗物质或尘埃标量场

Y_α : Conjugate momenta of dark matter or dust fields

Y_α : 暗物质或尘埃场的共轭动量

Λ : Cosmological constant

Λ : 宇宙学常数

$T_j, \underline{T}_J$: Respective Lie algebra basis

$T_j, \underline{T}_J$: 对应李代数基

$\tau_j, \underline{\tau}_J$: Respective Lie algebra basis matrix in defining representation

$\tau_j, \underline{\tau}_J$: 定义表示下的对应李代数基矩阵

∇ : Gauge-covariant differential annihilating E

∇ : 零化 E 的规范协变微分

$\mathcal{D}$: Gauge-covariant differential replacing Γ^j_a by A^j_a

$\mathcal{D}$: 将 Γ^j_a 替换为 A^j_a 的规范协变微分

We have chosen Lie algebra basis elements such that $\mathrm{Tr}\left(\tau_j\tau_k\right) = \delta_{jk}, \mathrm{Tr}\left(\underline{\tau}_J\underline{\tau}_K\right) = \delta_{JK}$ and thus do not need to pay attention to the position of these indices. If G is zero- or one-dimensional, we simply drop the index ι allowing to treat hypothetical scalar fields such as the inflaton. The differentials $\nabla, \mathcal{D}$ act on tensorial, spinorial, representation, and Lie algebra indices, i.e., for a hypothetical field $X^{aA}{}_{j\iota J}$ of density weight zero

我们选取的李代数基元满足 $\mathrm{Tr}\left(\tau_j\tau_k\right) = \delta_{jk}, \mathrm{Tr}\left(\underline{\tau}_J\underline{\tau}_K\right) = \delta_{JK}$ ，因此无需关注这些指标的位置。若 G 是零维或一维的，我们直接去掉指标 ι，即可处理暴胀子这类假设标量场。微分算符 $\nabla, \mathcal{D}$ 作用于张量指标、旋量指标、表示指标和李代数指标，即对于密度权为零的假设场 $X^{aA}{}_{j\iota J}$

$$\nabla_b X^{aA}_{j\iota J} = \partial_b X^{aA}_{j\iota J} + \Gamma^a_{bc} X^{cA}_{j\iota J} + \Gamma^k_b [T_k]_j{}^l X^{aA}_{l\iota J} + \Gamma^k_b [\tau_k]^A{}_B X^{aB}_{j\iota J}$$

$$+\underline{A}^K_b \left[\underline{\tau}_K\right]_\iota{}^\lambda X^{aA}_{j\lambda J} + \underline{A}^K_b \left[\underline{T}_K\right]_J{}^L X^{aA}_{j\iota L} \tag{1}$$

We are considering only one matter species of each kind; the case of several species possibly for different gauge groups can be obtained by simply adding those terms. In particular, fermions of opposite chirality can be treated by interchanging η^A_l with its adjoint $\left(\eta^A_l\right)^*$ which up to a factor of i plays the role of its conjugate momentum. Majorana fermions also can be treated like this; instead of having two independent chiralities $\eta, (\eta')^*$, we have $\eta' = \eta$.

我们仅考虑每类一种物质种类；多种类 (可能对应不同规范群) 的情况可直接通过添加对应项得到。具体而言，相反手征的费米子可通过交换 η^A_l 与其伴随 $\left(\eta^A_l\right)^*$ 处理，$\left(\eta^A_l\right)^*$ 相差因子 i 后即充当其共轭动量。马约拉纳费米子也可按此方式处理: 不存在两个独立手征 $\eta, (\eta')^*$ ，我们只有 $\eta' = \eta$ 。

From the quantum theory of detailed in → Chap. 83,"Emergence of Riemannian Quantum Geometry", we only need to know that there is a Hilbert space $\mathcal{H}$ with dense subset $\mathcal{D}$ consisting of the span of the orthonormal spin network function (SNWF) basis. A SNWF $T_{\gamma,j,\iota}$ is labeled by a triple γ, j, ι where γ is a piecewise (more precisely semi-)analytic, oriented graph with edge set $E(\gamma)$ and vertex set $V(\gamma)$, j is a set of irreducible representations j_e (spins and YM irreducibles for SU(2) x G) coloring the edges $e \in E(\gamma)$, and ι is a set of combinations of irreducible representations and gauge-invariant intertwiners ι_v coloring the vertices $v \in V(\gamma)$. The fields $A, \underline{A}$ are excited along the edges and the fields η, v, ϕ, X on the vertices.

根据 → 第 83 章“黎曼量子几何的涌现”中详述的量子理论，我们只需要知道存在希尔伯特空间 $\mathcal{H}$ ，其稠密子集 $\mathcal{D}$ 由正交归一自旋网络函数 (SNWF) 基的张成空间构成。一个自旋网络函数 $T_{\gamma,j,\iota}$ 由三元组 γ, j, ι 标记: 其中 γ 是分段 (更准确说是半解析) 的定向图，边集为 $E(\gamma)$ ，顶点集为 $V(\gamma)$, j ；j_e 是不可约表示集合 (SU(2)×G 对应的自旋和杨-米尔斯不可约表示)，给边 $e \in E(\gamma)$ 着色；ι 是不可约表示与规范不变交错算子 ι_v 的组合集合，给顶点 $v \in V(\gamma)$ 着色。场 $A, \underline{A}$ 沿边激发，场 η, v, ϕ, X 在顶点上激发。

Lagrangian, Legendre Transform, and Constraints

拉格朗日量、勒让德变换与约束条件

The Lagrangian for GR coupled to matter is a spacetime scalar density of weight one in order that the corresponding action is spacetime diffeomorphism invariant. Therefore, the Legendre transform with respect to a foliation $t \mapsto \sum_t$ of M yields constraints which are tensor densities of weight one. Thus, density weight one is the natural density weight which is dictated by general covariance. The leaves $\sum_t$ of the foliation are mutually spatially diffeomorphic by global hyperbolicity and thus diffeomorphic to some given 3-manifold σ . The fields listed in the previous subsection can be considered as fields on the manifold $\mathbb{R} \times \sigma$, and after performing the Legendre transform, the initial value constraints take the following form:

耦合物质的广义相对论拉格朗日量是权重为 1 的时空标量密度，这保证了对应的作用量具有时空微分同胚不变性。因此，对 $t \mapsto \sum_t$ 的叶状结构 M 做勒让德变换后，会得到同样是权重 1 张量密度的约束条件。可见，权重 1 是广义协变性要求的自然权重。由于整体双曲性，叶状结构的叶 $\sum_t$ 彼此之间空间微分同胚，因此都微分同胚于某个给定的 3 流形 σ 。上一小节列出的场可以视为定义在流形 $\mathbb{R} \times \sigma$ 上的场，完成勒让德变换后，初值约束具有如下形式：

$$G_j = \partial_a E_j^a + \varepsilon_{jkl} A_a^k E_l^a + J_j$$

$$\underline{G}_J = \partial_a \underline{E}_J^a + f_{JKL} \underline{A}_a^K \underline{E}_L^a + \underline{J}_J + \pi^T \underline{\tau}_J \phi$$

$$D_a = E_j^b \partial_a A_b^j - \partial_b \left(E_j^b A_a^j \right) + \underline{E}_J^b \partial_a \underline{A}_b^J - \partial_b \left(\underline{E}_J^b \underline{A}_a^J \right)$$

$$+ i\delta_{AB} \left[\left(\eta_\iota^A\right)^* \delta^{\iota\kappa} \partial_a \eta_\kappa^B + \left(\upsilon^A\right)^* \partial_a \upsilon^B - \text{ c.c. } \right] + \pi^\iota \partial_a \phi_\iota + Y_\alpha \partial_a X^\alpha$$

$$C = C_E^G + C_L^G + C^C + C^{YM} + C^F + C^S + C^Y + C^D$$

$$C_E^G = \frac{F_{ab}^j \varepsilon_{jkl} E_k^a E_l^b}{|\det(E)|^{1/2}}, \; C_L^G = 2\beta \frac{K_a^j K_b^l E_{[j}^a E_{l]}^b}{|\det(E)|^{1/2}}, \; C^C = \Lambda |\det(E)|^{1/2}$$

$$C^{YM} = \frac{1}{2} E_a^j E_b^k \delta_{jk} |\det(E)|^{1/2} \left[\underline{E}_J^a \underline{E}_K^b + \frac{1}{2} \varepsilon^{acd} \varepsilon^{bef} \underline{F}_{cd}^J \underline{F}_{ef}^K \right] \delta_{JK}$$

$$C^F = i \frac{E_j^a}{|\det(E)|^{1/2}} \left[\left(\eta^T\right)^* \tau_j \left(\nabla_a \eta\right) - \text{ c.c. } \right] + m\left(\eta, \eta^*\right)$$

$$C^S = \frac{1}{2} \frac{1}{|\det(E)|^{1/2}} \left\{ \left[\left(\pi^\iota\right)^*; \pi^\kappa + E_j^a E_k^b \delta^{jk} \left(\nabla_a \phi\right)_\iota^* \left(\nabla_b \phi\right)_\kappa \right] \delta^{\iota\kappa} \right.$$

$$\left. + |\det(E)| \, V\left(\left(\phi^T\right)^* \phi \right) \right\}$$

$$C^Y = \left[\left(\phi_\iota\right)^* \left(\upsilon^A\right)^* \delta_{AB} \delta_{\iota\kappa} \eta_\kappa^B + \text{ c.c } \right], \; C^D = h^D(X, Y, E) \tag{2}$$

Here, $F, \underline{F}$ are the curvature 2-forms of $A, \underline{A}$, respectively, $K_a^j := A_a^j - \Gamma_a^j$ is a derived field depending on both A, E closely related to the extrinsic curvature, β is a numerical real-valued and non-vanishing constant depending on the Immirzi parameter - Chap. 83, "Emergence of Riemannian Quantum Geometry" (which we set here to unity for simplicity of exposition), $J_j = \left(\eta^T\right) * \tau_j \eta, \underline{J}_J = \left(\eta^T\right) * \underline{\tau}_J \eta$ are purely fermionic currents bilinear in the fermion fields, $\varepsilon_{jkl} = (T_k)_{jl}, f_{JKL} = \left(\underline{T}_K\right)_{JL}$ are the respective structure constants, and $m(\eta, \eta^*)$ denotes a general, gauge-invariant bilinear mass term of Dirac and/or Majorana type and V a gauge-invariant polynomial potential (including mass terms). The suffix at the various contributions to C refers to the various geometry and matter species: Euclidian and Lorentzian geometry term, cosmological term, Yang-Mills term, fermionic term, scalar term, Yukawa term, and dark matter or dust term. To obtain the desired phenomenology, one can add more field species with corresponding mixing matrices and gauge groups. The common feature of the C^D contribution is that, in the models considered so far, it depends only on the dark matter or dust fields and the spatial geometry encoded by E . We also have set various coupling constants to unity for simplicity of presentation. See [97] for more details.

此处 $F, \underline{F}$ 分别是 $A, \underline{A}$ 的曲率 2-形式，$K_a^j := A_a^j - \Gamma_a^j$ 是同时依赖于 A, E 的导出场，与外曲率密切相关，β 是依赖于伊米尔齐参数的非零实数值常数——参见第 83 章“黎曼量子几何的涌现”(为简化表述我们这里将其取为 1)，$J_j = (\eta^T) * \tau_j \eta, \underline{J}_J = (\eta^T) * \underline{\tau}_J \eta$ 是费米子场的纯双线性费米流，$\varepsilon_{jkl} = (T_k)_{jl}, f_{JKL} = \left(\underline{T}_K\right)_{JL}$ 是相应的结构常数，$m(\eta, \eta^*)$ 表示狄拉克和/或马约拉纳型的一般规范不变双线性质量项，V 是规范不变多项式势 (包含质量项)。C 各项贡献的下标对应不同的几何与物质种类: 欧氏几何项、洛伦兹几何项、宇宙学项、杨-米尔斯项、费米子项、标量项、汤川项以及暗物质或尘埃项。为得到符合现象学的结果，还可以引入更多场种类，以及对应的混合矩阵和规范群。目前研究的模型中，C^D 贡献的共同特点是，它仅依赖于暗物质或尘埃场，以及由 E 编码的空间几何。为简化表述，我们同样将多个耦合常数都取为 1。更多细节参见文献 [97]。

The main reason for displaying these various contributions to the constraints explicitly is to draw attention to the following:

我们将约束条件的这些不同贡献明确写出，主要目的是为了强调以下内容:

1. Pure (Euclidian) vacuum GR is unphysical

1. 纯 (欧几里得) 真空广义相对论是非物理的

Except for the dark matter term and the inflaton terms, all pieces in the above list are experimentally confirmed, and the signature of the universe is Lorentzian and not Euclidian. Therefore, a theory of quantum geometry must ensure that all contributions to $G, \underline{G}, D, C$ are taken into account and not just bits and pieces of it.

除暗物质项和暴胀子项外，上述列表中的所有项都已得到实验证实，且宇宙的符号是洛伦兹型而非欧几里得型。因此，量子几何理论必须保证，对 $G, \underline{G}, D, C$ 的所有贡献都需被纳入考量，而非只考虑它的零散部分。

2. Non-polynomiality

2. 非多项式性

The non-polynomial character of the constraints is present only in the Hamiltonian constraint and is only due to the non-polynomial appearance of the field E_j^a which is a triad of density weight one. One may be tempted to multiply C by a sufficiently high power r of the density weight unity quantity

约束的非多项式特性仅存在于哈密顿约束中，且完全源于密度权为 1 的三元场 E^a_j 以非多项式形式出现。人们或许会想将 C 乘上足够高次幂的单位密度量 r

$$Q \;:=\; |\det(E)|^{1/2} \tag{3}$$

in order to turn it into a polynomial. In the classical theory, this is possible because here $Q > 0$ is an implicit assumption and thus the constraints $C, Q^r C$ define the same constraint surface. This is due to the fact that the relation to the intrinsic metric is given by

来将其转化为多项式。在经典理论中这是可行的，因为这里 $Q > 0$ 是默认成立的假设，因此约束 $C, Q^r C$ 定义的是同一个约束面，这源于它与内蕴度规的关系由下式给出

$$q^{ab}\det(q) \;:=\; E^a_j E^b_j \delta^{jk} \tag{4}$$

In the quantum theory, this equivalence is no longer manifest if $Q > 0$ is not granted also there. As we will see below, apart from that, quantum theory by itself dictates that one keeps the density unity constraint C as it is in its non-polynomial form.

在量子理论中，如果 $Q > 0$ 不被承认，这种等价性就不再成立。我们下文将会说明，除此之外，量子理论本身就要求我们保持密度权为 1 的约束 C 的非多项式原始形式。

3. Dictated density weights of the elementary fields

3. 基本场的既定密度权重

The explicit form of D_a shows that the various fields have the following density weights that naturally come out of the Legendre transform: $A, \underline{A}, \phi, X$ have density weight zero, $E, \underline{E}, \pi, Y$ have density weight one, and η, υ have density weight 1/2 .

D_a 的显式形式表明，勒让德变换自然导出各类场具有如下密度权重：$A, \underline{A}, \phi, X$ 的密度权重为零，$E, \underline{E}, \pi, Y$ 的密度权重为 1，η, υ 的密度权重为 1/2 。

4. Universal coupling of E

4. E 的普适耦合

We see explicitly that the field E appears in every single term of the Hamiltonian constraint; it couples to A and matter in various different forms:

我们可以清楚看到，场 E 出现在哈密顿约束的每一项中；它以多种不同形式与 A 和物质耦合:

$$\frac{\epsilon^{jkl}E^a_k E^b_l}{|\det(E)|^{1/2}}, \delta_{jk}E^j_a E^k_b, \frac{E^a_j}{|\det(E)|^{1/2}}, \frac{1}{|\det(E)|^{1/2}}, |\det(E)|^{1/2}, \frac{E^a_j E^b_k \delta_{jk}}{|\det(E)|^{1/2}},$$

(5)

Here, the field E_a^j is the inverse of E_j^a , i.e., $E_j^a E_a^k = \delta_j^k$. The list (2) is incomplete because we have to remember that $K_a^j = A_a^j + \Gamma_a^j$ and Γ_a^j , which also appears in the fermionic $\nabla\eta$, is of the form $\frac{E\cdot E\cdot\partial E}{\det(E)}$. This again shows that it is a necessary assumption in the classical theory that E be non-degenerate, in particular, $Q := |\det(E)|^{1/2} > 0$ everywhere and every time.

此处，场 E_a^j 是 E_j^a 的逆，即 $E_j^a E_a^k = \delta_j^k$ 。上述列表 (2) 并不完整，因为我们需要注意，同样出现在费米子 $\nabla\eta$ 中的 $K_a^j = A_a^j + \Gamma_a^j$ 和 Γ_a^j 形如 $\frac{E\cdot E\cdot\partial E}{\det(E)}$ 。这再次说明，经典理论中必须假定 E 是非退化的，具体而言，要求 $Q := |\det(E)|^{1/2} > 0$ 在任何时间、任何位置都成立。

Poisson Algebra of Constraints

约束的泊松代数

As follows from the abstract argument in [67] and as one can also verify by tedious calculation using the canonical brackets between the list of fields in (1), the constraints of the previous subsection obey the subsequent universal (i.e., independent of the Lagrangian) Poisson algebra

正如文献 [67] 中的抽象论证所得出的结论，同时也可以通过对 (1) 中场列表之间的正则括号进行繁琐计算验证，上一小节的约束满足下述通用 (即独立于拉格朗日量的) 泊松代数

$$\{(G,\underline{G})[L,\underline{L}],(G,\underline{G})[L',\underline{L}'],=-(G,\underline{G})[[L,L'],[\underline{L},\underline{L}']]$$

$$\{(G,\underline{G})[L,\underline{L}],D[u]\}=-\{(G,\underline{G})[u[L],u[\underline{L}]]$$

$$\{(G,\underline{G})\left[f,\underline{f}\right],C[f]\}=0$$

$$\{D[u],D[u']\}=-D[[u,u']]$$

$$\{D[u],C[f]\}=-C[u[f]]$$

$$\{C[f],C[f']\}=-D\left[q^{-1}[fdf'-f'df]\right] \tag{6}$$

where

其中

(G,

$$(G,\underline{G})[L,\underline{L}]=\int_\sigma d^3\left[L^jG_j+\underline{L}^J\underline{G}_J\right],D[u]=\int_\sigma d^3u^aD_a,C[f]=\int_\sigma d^3fC$$

(7)

are the smeared constraints. Here, $[L, L']$ denotes the commutator in the Lie algebra of $\mathrm{SU}(2)$, and similar for the group G, $[u, u']$ is the commutator in the Lie algebra of vector fields, and $u[f]$ is the action of the derivation u on the scalar f .

是弥散约束。此处 $[L, L']$ 表示 $\mathrm{SU}(2)$ 的李代数中的对易子，类似地，群 G, $[u, u']$ 对应向量场李代数中的对易子，而 $u[f]$ 是导子 u 在标量 f 上的作用。

The subalgebra spanned by the $D[u]$, $C[f]$ alone is known as hypersurface deformation algebra, and it is not a Lie algebra due to the presence of the field q^{-1} displayed in (4) in the structure functions.

仅由 $D[u]$, $C[f]$ 张成的子代数称为超曲面形变代数，由于结构函数中出现了 (4) 所示的场 q^{-1} ，它不是一个李代数。

Kinematical Hilbert Space Representations

运动学希尔伯特空间表示

Although in Chap. 83, "Emergence of Riemannian Quantum Geometry" an elegant argument based on diffeomorphism covariance is presented which establishes that the kinematical Hilbert space representation is essentially unique, we give here a shorter argument based on the dynamics of the theory displayed above.

尽管在第 83 章“黎曼量子几何的涌现”中，已经基于微分同胚协变性给出了一个优美的论证，证明运动学希尔伯特空间表示本质上是唯一的, 但我们在此将基于前文展示的理论动力学给出一个更简短的论证。

In QFT on Minkowski space, one usually splits the polynomial Hamiltonian into quadratic and higher-order part and uses the quadratic part to select a (Fock) vacuum. Then the quadratic part is densely defined on the resulting free or "kinematical" Fock space. The chosen Fock vacuum also defines what one means by normal ordering and in that sense also severely affects the higher-order part. In our non-perturbative and non-polynomial setting, a natural split into quadratic and higher order is not available. However, the fact that the field E couples to every single term in the Hamiltonian constraint in algebraically different forms strongly motivates to select a vacuum Ω annihilated by E in order that each of those terms be individually densely defined. This also requires inverse powers of $Q(3)$ to annihilate the vacuum which is possible as we will see.

在闵氏空间量子场论中，人们通常将多项式哈密顿量拆分为二次项和高阶项，利用二次项选取一个 (福克) 真空。随后二次项在得到的自由即“运动学”福克空间上是稠定的。所选的福克真空还定义了正规序的含义，从这个角度来说也会对高阶项产生极大影响。在我们的非微扰、非多项式框架下，无法自然地拆分出二次项和高阶项。然而，场 E 以代数上不同的形式耦合到哈密顿约束的每一项，这一事实强烈支持我们选取一个被 E 湮灭的真空 Ω ，从而使得每一项各自都是稠定的。这也要求 $Q(3)$ 的逆幂湮灭真空，我们稍后会看到这是可以实现的。

We now show that the innocent-looking condition $E_j^a \Omega = 0$ has far-reaching consequences [100]. As it is customary in QFT, instead of considering the abstract Heisenberg algebra generated by the relations (exemplified for geometry)

我们现在将证明，看似简单的条件 $E_j^a\Omega = 0$ 拥有影响深远的结论 [100]。和量子场论中的常规做法一样，我们不考虑由 (几何示例的) 关系生成的抽象海森堡代数，

$$[E[f], E[g]] = [F[A], F'[A]] = 0, [E[f], F[A]] = i\mathbf{1}, E[f]^* - E[f]$$

$$= F[A]^* - F[A] = 0 \tag{8}$$

with $F[A] := \int_\sigma d^3x F_j^a A_a^j, E[f] := \int_\sigma d^3x f_a^j E_j^a$ where f, F are real-valued test functions, we consider the Weyl algebra generated by Weyl elements $U[F] = \exp(iF[A]), V[f] = \exp(iE[f])$ and induced relations

结合 $F[A] := \int_\sigma d^3x F_j^a A_a^j, E[f] := \int_\sigma d^3x f_a^j E_j^a$ (其中 f, F 是实值测试函数)，我们转而考虑由外尔元 $U[F] = \exp(iF[A]), V[f] = \exp(iE[f])$ 生成的外尔代数以及导出关系

$$U[F]\,U[F'] = U[F+F'], U[F]^* = U[-F], V[f]\,V[f'] = V[f+f'],$$

$$V[f]^* = V[-f]\,V[f]\,U(F)\,V[-f] = U[F]\,e^{-iF[f]} \tag{9}$$

In any representation, the Weyl elements are unitary, i.e., bounded operators so that no domain questions arise as compared to the generators of the Heisenberg algebra. It now follows from our assumption $V[f]\Omega = \Omega$ for all f that

在任何表示中，外尔元都是幺正有界算子，因此相比海森堡代数的生成元，不存在定义域问题。从我们对所有 f 成立的假设 $V[f]\Omega = \Omega$ 可以推出

$$< \Omega, U[F]\Omega > = < V[-f]\Omega, U[F]V[-f]\Omega > = e^{-iF[f]} < \Omega, U[F]\Omega >$$

(10)

for all f, F . It follows that the representation is necessarily of Narnhofer-Thirring type [77]

对所有 f, F 成立。由此可知该表示必然属于 Narnhofer-Thirring 类型 [77]

$$< \Omega, U[F]\Omega > = \delta_{F,0} \tag{11}$$

where δ really means the Kronecker function. This means that the unitary operators $U[F]$ are not strongly or weakly continuous and that the states $U[F]\Omega$ are orthonormal implying that the resulting Hilbert space spanned by the $U[F]\Omega$ is not separable.

其中 δ 实际指克罗内克函数。这意味着幺正算子 $U[F]$ 不强连续也不弱连续，且态 $U[F]\Omega$ 是正交归一的，说明由 $U[F]\Omega$ 张成的希尔伯特空间是不可分的。

The LQG representation uses more complicated functions than the $U[F]$ based on holonomies of A along one-dimensional curves as it is motivated by Gauß gauge covariance, but the main argument above still applies. We will see in a moment that besides gauge covariance, again purely dynamical arguments dictate that F should smear A in one dimension and that f should smear E in two dimensions.

受高斯规范协变性的启发，LQG 表示所使用的函数比基于 $U[F]$ 的函数更复杂，它采用 A 沿一维曲线的和乐，但上述核心论证仍然成立。我们很快就会看到，除规范协变性外，纯动力学论证也同样要求 F 对 A 做一维涂抹，要求 f 对 E 做二维涂抹。

As far as the matter representations are concerned, motivated by background independence which excludes Fock-type representations, one can similarly pick representations defined by the requirements $\underline{E}\Omega = \pi\Omega = \eta\Omega = \upsilon\Omega = Y\Omega = 0$ [98]. For the YM, scalar, and DM sector, the corresponding representations are again discontinuous, and the smearing dimensions in the YM sector are the same as in the geometry sector, while for the scalar sector, ϕ, π and X, Y are smeared in zero and three dimensions, respectively. We can therefore pick a Hilbert space representation for the YM sector analogous to the geometry sector and for the scalar sector an analogous representation based on ”point holonomies.” For the fermionic sector, we obtain the usual continuous Fock representation familiar from the standard model quantization, and the natural smearing dimension is 3/2. In this way, scalar and fermionic matter is excited at points, while geometry and YM degrees of freedom are excited along paths. A combined generalized SNWF for all degrees of freedom is thus labeled by a graph with irreducible representations of both SU(2) and G labeling the edges corresponding to the geometry and YM field excitations, irreducible representations for SU(2) at the vertices for the fermions, and irreducible representations at the vertices for the fermions and Higgs scalars as well as gauge-invariant intertwiners at the vertices; see [98] for details and section ”Notation”.

就物质表示而言，在排除福克型表示的背景无关性驱动下，我们同样可以选取由条件 $\underline{E}\Omega = \pi\Omega = \eta\Omega = \upsilon\Omega = Y\Omega = 0$ 定义的表示 [98]。对于杨-米尔斯、标量和暗物质部分，相应表示同样是不连续的，杨-米尔斯部分的涂抹维度与几何部分一致，而标量部分中，ϕ, π 和 X, Y 分别在零维与三维涂抹。因此我们可以为杨-米尔斯部分选取类比几何部分的希尔伯特空间表示，为标量部分选取基于“点和乐”的类似表示。对于费米子部分，我们得到标准模型量子化中常见的常规连续福克表示，其自然涂抹维度为 3/2。通过这种方式，标量和费米子物质在点上激发，而几何和杨-米尔斯自由度沿路径激发。因此，所有自由度的组合广义自旋网函数基由图标记: 该图的边对应几何和杨-米尔斯场激发，分别标记 SU(2) 和规范群 G 的不可约表示；顶点处标记费米子对应的 SU(2) 不可约表示，还标记费米子和希格斯标量的不可约表示，以及顶点处规范不变交缠元；详见 [98] 和“记号”一节。

We will see below that also those representations and smearing dimensions are naturally suggested by the dynamics.

我们下文将会看到，这些表示和涂抹维度也同样由动力学自然给出。

Smearing Dimensions and Density Weights

涂抹维度与密度权重

The main mathematical problem of QFT is to define products of operator-valued distributions. In the density weight one form, the problem in quantum gravity is even worse, as one also has to deal with inverse powers of fields. Yet, it is precisely that the non-polynomial nature of the constraints offers the possibility to arrive at a non-perturbative quantization.

量子场论的核心数学问题是定义算子值分布的乘积。在密度权重为 1 的形式中，量子引力中的这个问题更为棘手，因为我们还必须处理场的逆幂次。不过，恰恰是约束的非多项式性质为非微扰量子化提供了可能。

As it is customary in QFT, we try to tame the products of distributions by introducing a short-distance cut-off ε . This is conveniently obtained using a, e.g., simplicial decomposition $\mathcal{T}$ of σ into tetrahedra Δ of coordinate volume of order ε^3 . The classical Hamiltonian constraints can now be written as

按照量子场论中的常规做法，我们通过引入短距离截断 ε 来正则化分布的乘积。这可以方便地通过例如将 σ 单纯分解 $\mathcal{T}$ 为坐标体积量级为 ε^3 的四面体 Δ 来实现。经典哈密顿约束现在可以写为

$$C_{\varepsilon}[f]=\sum_{\Delta\in\mathcal{T}} C_{\varepsilon}^{\Delta}[f], C_{\varepsilon}^{\Delta}[f]=\int_{\Delta} d^3x f C \tag{12}$$

which is still exact. If p_Δ denotes the barycenter of Δ in the chosen system of coordinates, then we have (up to numerical constants)

上述形式仍然是精确的。如果 p_Δ 表示所选坐标系中 Δ 的重心，那么我们可得 (数值常数不计)

$$C_{\varepsilon}^{\Delta}[f]\approx f(p_\Delta)\left[C(p_\Delta)\varepsilon^3\right] \tag{13}$$

The challenge is now to approximate the continuum fields that appear in $C(p_\Delta)\varepsilon^3$ by smeared versions that have well-defined operator substitutes on the kinematical Hilbert space such that the following holds:

目前的挑战是用涂抹版本近似 $C(p_\Delta)\varepsilon^3$ 中出现的连续场，使得涂抹后的场在运动学希尔伯特空间上具有定义良好的算子替代，并满足以下条件:

1. The resulting classical expression $\widehat{C}_{\varepsilon}^{\Delta}[f]$ differs from $C(p_\Delta)\varepsilon^3$ by a term of order ε^4 .

1. 最终得到的经典表达式 $\widehat{C}_{\varepsilon}^{\Delta}[f]$ 与 $C(p_\Delta)\varepsilon^3$ 的差别为一个量级为 ε^4 的项。

2. When the operator substitution in $\widehat{C}_{\varepsilon}^{\Delta}[f]$ is carried out, the operator $\widehat{C}_{\varepsilon}[f]=\sum_{\Delta}\widehat{C}_{\varepsilon}^{\Delta}[f]$ is densely defined on the kinematical Hilbert space with dense, invariant domain independent of f .

2. 完成 $\widehat{C}_{\varepsilon}^{\Delta}[f]$ 中的算子替换后，算子 $\widehat{C}_{\varepsilon}[f]=\sum_{\Delta}\widehat{C}_{\varepsilon}^{\Delta}[f]$ 在运动学希尔伯特空间上是稠定的，其稠密不变定义域不依赖于 f 。

3. The limit $\varepsilon\to 0$ of $\widehat{C}_{\varepsilon}[f]\psi, \psi\in D$ exists and is not trivial for every $\psi\in\mathcal{D}$.

3. $\widehat{C}_{\varepsilon}[f]\psi, \psi\in D$ 的极限 $\varepsilon\to 0$ 存在，且对任意 $\psi\in\mathcal{D}$ 都非平凡。

Requirement 1 just makes sure that the classical $\widehat{C}_{\varepsilon}^{\Delta}[f]$ is an admissible discretiza-tion. Requirement 2 is the main point of this point splitting regularization, namely, to give the regulated expression a mathematical meaning. Insisting that the domain is invariant and independent of f is to enable computing commutators. Requirement 3 encodes the removal of the regulator defining the regulator-independent operator densely on $\mathcal{D}$ and such that it is not the trivial, zero operator.

要求 1 仅用于保证经典 $\widehat{C}_{\varepsilon}^{\Delta}[f]$ 是可容许的离散化。要求 2 是这种点分裂正则化的核心目标，即为正则化后的表达式赋予数学意义。要求定义域不变且不依赖于 f 是为了能够计算对易子。要求 3 体现了正则化因子的移除，用以在 $\mathcal{D}$ 上定义不依赖于正则化的稠定算子，且保证该算子不是平凡的零算子。

These requirements together with the concrete form of the Hamiltonian constraint now uniquely fix the (integer) smearing dimensions of the elementary fields as well as that changing C to $Q^r C$ for some real number $r \neq 0$ is not allowed. The key is to realize that the factor ε^3 universally multiplies every single term of $C(p_\Delta)$, i.e., when rewriting the "naked" fields at the point p_Δ in terms of their smeared versions, the factor ε^3 must be precisely absorbed by every single matter and geometry contribution.

这些要求结合哈密顿约束的具体形式，现在可以唯一确定基本场的 (整数) 涂抹维度，并且不允许将 C 更改为 $Q^r C$ (对任意实数 $r \neq 0$)。关键在于认识到因子 ε^3 会整体乘在 $C(p_\Delta)$ 的每一项上，也就是说，当我们将点 p_Δ 处的“裸”场改写为其涂抹版本时，因子 ε^3 必须恰好被所有物质和几何贡献完全吸收。

We begin with the cosmological term with density weight w . In order that $Q^w(p_\Delta)\varepsilon^3$ satisfies requirements 2 and 3, the expression $\varepsilon^{2/w}E_j^a(p_\Delta)$ must be substitutable by a well-defined smeared field operator; hence, the smearing dimension of E is $2/w$.

我们从密度权重为 w 的宇宙学项开始讨论。为了让 $Q^w(p_\Delta)\varepsilon^3$ 满足要求 2 和 3，表达式 $\varepsilon^{2/w}E_j^a(p_\Delta)$ 必须能够被定义良好的涂抹场算子替代；因此，E 的涂抹维度为 $2/w$ 。

Next we consider the spatial curvature term which is implicitly contained in C_L^G and is given by $\varepsilon^3 Q^{-1}(\Gamma_a \times \Gamma_b)\cdot\left(E^a \times E^b\right)$ plus a similar term with Γ^2 replaced by $\partial\Gamma$. From the definition of the spin connection

接下来我们考虑隐含在 C_L^G 中的空间曲率项，它由 $\varepsilon^3 Q^{-1}(\Gamma_a \times \Gamma_b)\cdot\left(E^a \times E^b\right)$ 加上一个将 Γ^2 替换为 $\partial\Gamma$ 的相似项给出。根据自旋联络的定义

$$(\nabla_a E)_j^b = \left(\nabla_a E_j\right)^b + \varepsilon_{jkl}\Gamma_a^k E_l^b = 0 \tag{14}$$

where the first term just acts on the tensor structure of E , we see that Γ_a^j is schematically of the form $Q^{-2}E\ E\ \partial E$. Thus, the spatial curvature term is schematically of the form $\varepsilon^3 Q^{-5}E^6(\partial E)^2 = \frac{\left[\varepsilon^3\partial E\right]^2}{\varepsilon^3 Q}$. As $Q\varepsilon^3$ is already well defined by the paragraph before (14), $\varepsilon^3\partial E$ must be well defined. As ∂ increases the required smearing dimension by one unit (integrate by parts to see this), $\varepsilon^2 E$ must be well defined, i.e., the smearing dimension of E is 2 . It follows unambiguously that $w = 1$ and that the above-discussed similar term is also well defined.

其中第一项仅作用于 E 的张量结构，我们可知 Γ_a^j 的概形如 $Q^{-2}E\ E\ \partial E$ 。因此，空间曲率项的概形为 $\varepsilon^3 Q^{-5}E^6(\partial E)^2 = \frac{\left[\varepsilon^3\partial E\right]^2}{\varepsilon^3 Q}$ 。由于 $Q\varepsilon^3$ 已通过 (14) 前一段落良好定义，$\varepsilon^3\partial E$ 必须是良好定义的。由于 ∂ 会将所需弥散维度增加一个单位 (分部积分即可验证)，$\varepsilon^2 E$ 必须是良好定义的，即 E 的弥散维度为 2。由此可以明确推出 $w = 1$ ，且上述讨论的相似项也是良好定义的。

Next we consider the Euclidian geometry term C_E^G at density weight $w = 1$ which in particular contains a term of the schematic form $\varepsilon^3\left(AAEEQ^{-1}\right)(p_\Delta)$. The factor ε^3 must then be distributed as $(A\varepsilon)^2\left(E\varepsilon^2\right)^2\left(Q\varepsilon^3\right)^{-1}$

, i.e., the smearing dimension of A must be unity.

接下来我们考虑密度权重为 $w = 1$ 的欧几里得几何项 C_E^G ，它尤其包含一个概形为 $\varepsilon^3\left(AAEEQ^{-1}\right)(p_\Delta)$ 的项。因子 ε^3 必须按 $(A\varepsilon)^2\left(E\varepsilon^2\right)^2\left(Q\varepsilon^3\right)^{-1}$ 分配，即 A 的弥散维度必须为 1。

Continuing this reasoning in the various matter contributions to C , we find unambiguously that $(\underline{A},\underline{E})$ have smearing dimension 1,2, respectively; that (ϕ,π) and (X,Y) have smearing dimension 0,3, respectively; and that (η,υ) have smearing dimension $\frac{3}{2}$, respectively. The fact that all of this fits together although the algebraic form of the various terms in C is quite different is quite remarkable and is due to the diffeomorphism covariance of the theory.

将该推理推广到对 C 的各类物质贡献，我们可以明确推出: $(\underline{A},\underline{E})$ 的弥散维度分别为 1 和 2；(ϕ,π) 和 (X,Y) 的弥散维度分别为 0 和 3；(η,υ) 的弥散维度分别为 $\frac{3}{2}$ 。尽管 C 中各项的代数形式差异很大，但所有结果都自洽，这一事实十分值得注意，它源于该理论的微分同胚协变性。

Inverse Powers of E and Quantization Ambiguities

E 的逆幂与量子化歧义

From the form of C , it is clear that inverse powers of E only appear in the form $\left[\varepsilon^3 Q\right]^{-n}$. In order to make these well defined, one should make sure that (1) the volume of regions R given classically by $V(R) = \int_R d^3xQ$ becomes a well-defined operator and (2) that $V(R)^{-n}$ can be given a meaning. The first task is precisely accomplished in the chosen representation [6, 7, 86] as reviewed in ▶ Chap. 83, ”Emergence of Riemannian Quantum Geometry”.

由 C 的形式可知，E 的逆幂仅以 $\left[\varepsilon^3 Q\right]^{-n}$ 的形式出现。为了让这些表达式定义良好，必须确保:(1) 经典中由 $V(R) = \int_R d^3xQ$ 给出的区域 R 的体积成为一个良定义算符，(2) $V(R)^{-n}$ 可以被赋予意义。第一项任务恰好可在选定表示 [6, 7, 86] 中完成，正如 ▶ 第 83 章“黎曼量子几何的涌现”中所评述的。

The second task can be accomplished in many different ways; see [97] for a complete classification. The first possibility rests on the Poisson bracket identity

第二项任务可以通过多种不同方式完成；完整分类参见文献 [97]。第一种可能性基于泊松括号恒等式

$$\left\{V(R), A_a^j(x)\right\} = \left[QE_a^j\right](x) \Rightarrow \det\left(\{V(R), A(x)\}\right) = Q \tag{15}$$

for any $x \in R$. Thus, for any region of coordinate size ε^3 ,

对任意 $x \in R$ 成立。因此，对任意坐标尺寸为 ε^3 的区域,

$$V(R)^{-n} = V(R)^{-[n+m]}\left[\det\left(\{V(R), A_e(p)\}\right]^m\right.$$

$$= \frac{1}{r^3}\left[\left\{\det\left(V(R)^r, A(e)\right\}\right]^m, 3(r-1)m = -(m+n) \tag{16}$$

where $p \in R, e = (e_1, e_2, e_3)$ are the axes of a local coordinate system in R and $A^j_{e_I}(p) = \int_{e_I \cap R} A^j, I = 1, 2, 3$. The equality (16) is modulo terms of order higher than ε^3 . For any m such that $(m+n)/(3m) < 1$, i.e., $m > n/2$, we have $0 < r < 1$; thus, the r.h.s. of (16) depends on positive powers of $V(R)$. Since also to order ε we have $\mathrm{Tr}\left(h_{e_I}(A)\tau_j\right) = A^j_{e_I}(p)$ where h_e denotes the holonomy of A along e , we can turn (16) into a well-defined operator by replacing the classical objects $V(R), h_e$ by their operator equivalents and the Poisson brackets into commutators divided by i (if we set $\hbar = 1$) by using the spectral theorem on the self-adjoint positive operator $V(R)$.

其中 $p \in R, e = (e_1, e_2, e_3)$ 是 R 中局部坐标系的坐标轴，且 $A^j_{e_I}(p) = \int_{e_I \cap R} A^j, I = 1, 2, 3$ 。等式 (16) 模去 ε^3 阶以上的项。对任意满足 $(m+n)/(3m) < 1$ 即 $m > n/2$ 的 m ，我们有 $0 < r < 1$ ；因此，(16) 的右侧依赖于 $V(R)$ 的正幂次。此外，由于到 ε 阶我们有 $\mathrm{Tr}\left(h_{e_I}(A)\tau_j\right) = A^j_{e_I}(p)$ ，其中 h_e 表示 A 沿 e 的和乐，我们可以通过如下方式将 (16) 转化为良定义算符: 将经典对象 $V(R), h_e$ 替换为对应的算符，把泊松括号替换为除以 i 的对易子 (若我们设定 $\hbar = 1$)，再对自伴正算符 $V(R)$ 应用谱定理。

The second possibility is the Tikhonov regularization [118]

第二种方法是吉洪诺夫正则化 [118]

$$V(R)^{-n} := \lim_{\delta \to 0}\left[\frac{V(R)}{\delta^2 + V(R)^2}\right]^n \tag{17}$$

Common to both possibilities is that with this definition, and if one orders the dependence on the volume operator to the outmost right, every contribution to C only acts in the vicinity of a vertex of a SNWF T_γ over γ because $V(R)\,T_\gamma = \sum_{v \in V(\gamma) \cap R} V_v T_\gamma$ where $V(\gamma)$ is the set of at least tri-valent (with respect to the geometry degrees of freedom) vertices of γ and V_v a densely defined local operator. Thus, only those tetrahedra Δ of the simplicial decomposition contribute which contain a tri-valent vertex of γ . It is then natural to adapt the simplicial decomposition to γ so that $v = p_\Delta$ is that vertex if Δ contains one (for $\mathcal{T}$ sufficiently fine, each Δ contains at most one) and the edges e_I appear as segments of edges of γ adjacent to v . See [92, 94] for more details. That the action of C will eventually restrict to the vertices of γ will make sure that the resulting operator is densely defined in the SNWF basis. Again, it is quite remarkable that this is possible at all, given the tremendous degree of non-polynomiality of the constraint C .

两种可能性的共同点是: 采用该定义，若将体积算符的依赖项排序到最右侧，则对 C 的所有贡献仅作用于 SNWF T_γ 在 γ 上的一个顶点邻域，原因是 $V(R)\,T_\gamma = \sum_{v \in V(\gamma) \cap R} V_v T_\gamma$ ，其中 $V(\gamma)$ 是 γ 中 (就几何自由度而言) 至少为三价的顶点集合，V_v 是一个稠定局部算符。因此，只有单纯分解中包含 γ 三价顶点的那些四面体 Δ 会产生贡献。自然可将单纯分解适配于 γ ，从而若 Δ 包含一个三价顶点，则该顶点就是 $v = p_\Delta$ (当分解 $\mathcal{T}$ 足够精细时，每个 Δ 至多包含一个)，且边 e_I 是 γ 中与 v 相邻的边的分段。更多细节参见 [92, 94] 。C 的作用最终局限于 γ 的顶点，这保证了得到的算符在 SNWF 基下是稠定的。考虑到约束 C 具有极高的非多项式性，这一结论能够成立本身就十分值得注意。

We note that inverse volume powers give rise to a substantial amount of regularization ambiguities in addition to ordering ambiguities: First because we may pick any $m > n/2$ or may choose to write $n = n_1 + .. + n_r$

and treat each inverse power $n_k, k = 1, .., r$ individually and because instead of picking the spin 1/2 representation in the approximation $\mathrm{Tr}\left(h_{e_I}(A)\tau_j\right) = A^j(e_I)$ we may pick any spin k rep $\mathrm{Tr}\left(\pi_k\left(h_{e_I}(A)\right)\tau_j^{(k)}\right) = d_k A^j(e_I)$ where $d_k = 2k+1$ is its dimension and $\tau_j^{(k)}$ the Lie algebra basis element in that representation [80]. These ambiguities arise because the constraints are not polynomial so that the ”principle of simplicity” appears less natural than in regularizations of usual QFTs.

我们注意到,除排序歧义外,体积逆幂还会引发大量正则化歧义:首先,我们可以任选 $m > n/2$,也可以选择写成 $n = n_1 + .. + n_r$ 并单独处理每个逆幂 $n_k, k = 1, .., r$;此外,在近似 $\mathrm{Tr}\left(h_{e_I}(A)\tau_j\right) = A^j(e_I)$ 中我们不必选取自旋 1/2 表示,也可以任选自旋 k rep $\mathrm{Tr}\left(\pi_k\left(h_{e_I}(A)\right)\tau_j^{(k)}\right) = d_k A^j(e_I)$,其中 $d_k = 2k+1$ 是表示的维数,$\tau_j^{(k)}$ 是该表示下的李代数基元 [80]。这些歧义的来源是约束并非多项式,因此“简单性原则”在这里不如在普通量子场论正则化中自然。

Complete Regulated Operator

完全正则化算子

We sketch the regularization of the individual terms of C referring to [97] for details.

我们概述 C 各项的正则化,细节参见文献 [97]。

The piece C_E^G requires the quantization of F . Since A or F do not exist as operator-valued distributions in this representation (due to the discontinuity of Weyl elements mentioned in section ”Kinematical Hilbert Space Representations”), one replaces $\varepsilon^2 F$ by the holonomy along a loop starting in a vertex v of γ along the segments of two adjacent edges of γ whose endpoints are connected by a new edge. This is why the action is called graph changing. This involves a sum over pairs of such edges. In [8, 75] , this ”loop attachment” is chosen not along existing edges but ”close but disjoint” from those. This has the advantage of a simpler solution structure but does not lead to propagation [115] in the space of solutions.

其中 C_E^G 部分需要对 F 做量子化。由于在该表示中,A 或 F 都不能作为算子值分布存在 (原因是“运动学希尔伯特空间表示”章节提到的外尔元素不连续),因此我们将 $\varepsilon^2 F$ 替换为沿闭合圈的全纯性:该闭合圈起始于 γ 的顶点 v,沿 γ 两条相邻边的线段延伸,端点由一条新边连接。因此该作用被称为图变换作用。这一步需要对所有这类边对求和。在 [8, 75] 中,这种“加圈”操作并不沿现有边选取,而是选取与现有边“接近但不相交”的路径。这种做法的优势是解结构更简单,但无法在解空间中产生传播 [115]。

The piece C_L^G requires the quantization of $\varepsilon K_a^j = \varepsilon\left(A_a^j - \Gamma_a^j\right)$. This can be done using the classical Poisson bracket identity

C_L^G 部分需要对 $\varepsilon K_a^j = \varepsilon\left(A_a^j - \Gamma_a^j\right)$ 做量子化,可以利用经典泊松括号恒等式完成

$$\varepsilon K_a^j = \left\{\left\{C_E^G[1], V(\sigma)\right\}, \varepsilon A_a^j\right\} \tag{18}$$

and replace classical objects by quantum counterparts and Poisson brackets by commutators divided by i (also εA is substituted by a holonomy) using that C_E^G is already defined. Another possibility is based on quantizing the spin connection Γ itself [2].

再将经典对象替换为量子对应物，泊松括号替换为除以 i 的对易子 (同时将 εA 替换为全纯性)，利用这一方法处理时 C_E^G 已经是定义好的量。另一种可行方法是直接对自旋联络 Γ 本身做量子化 [2]。

The piece C^C is essentially $V(\sigma)$ which is already quantized.

C^C 部分本质上就是已经完成量子化的 $V(\sigma)$ 。

The piece C^{YM} replaces $\varepsilon^2\underline{E}$ by a YM flux operator and $\varepsilon^2\underline{F}$ by YM loop holonomies in complete analogy to (E,A), while the E -dependent terms are treated according to section "Inverse Powers of E and Quantization Ambiguities".

C^{YM} 部分将 $\varepsilon^2\underline{E}$ 替换为杨-米尔斯通量算子，将 $\varepsilon^2\underline{F}$ 替换为杨-米尔斯圈全纯性，整个过程和 (E,A) 完全类似，而依赖 E 的项则按照“E 的逆幂次与量子化歧义”章节的方法处理。

The piece C^F replaces $\varepsilon^{3/2}(\eta,v)$ by Fock operators located at vertices and otherwise treats the ingredients E,A,K as before.

C^F 部分将 $\varepsilon^{3/2}(\eta,v)$ 替换为位于顶点的福克算子，其余分量 E,A,K 仍按照之前的方法处理。

The piece C^S replaces $\varepsilon^3\pi(v)$ by i times an ordinary derivative w.r.t. $\phi(v)$, while $2\varepsilon[\partial_a\phi](v)$ is replaced by $\left[e^{i\phi(v+\varepsilon\delta_a)}-e^{i\phi(v-\varepsilon\delta_a)}\right]e^{-i\phi(v)}$ in case that ϕ transforms in the trivial or adjoint representation (if it transforms in the defining representation as in the SM, one should first carry out the Higgs mechanism classically and reduce the treatment to the trivial representation) where δ_a denotes a translation in local a-direction and $\phi(v)-\phi(v_0)$ itself is replaced by an ε resolution Riemann sum approximation of $i^{-1}\int_{c_v}dW(x)W(x)$, $W(x)=\exp(i\phi(x))$ where c_v is a path from some reference point v_0, at which ϕ decays to zero or is otherwise fixed by the classical boundary conditions, to v. Alternatively, one may instead work with the more general functions $W_\mu(x)=e^{i\mu\phi(x)}$ and replace $\phi(x)$ by $\left[W_\mu(x)-W_{-\mu}(x)\right]/(2i\mu)$ for fixed μ [12]. The then introduced additional dependence of the operator on μ is much debated in the loop quantum cosmology (LQC) literature [1, 9,40] (see also - Chaps. 90, "Loop Quantum Cosmology: Relation Between Theory and Observations" and B89, "Loop Quantum Cosmology: Physics of Singularity Resolution and Its Implications") which focuses on the cosmological sector of the theory and uses a lot of the technology introduced above for the full theory.

项 C^S 将 $\varepsilon^3\pi(v)$ 替换为 i 乘以对 $\phi(v)$ 的普通导数，当 ϕ 属于平凡表示或伴随表示时 (若 ϕ 如标准模型中那样属于定义表示，则应先经典地执行希格斯机制，将处理约化到平凡表示)，$2\varepsilon[\partial_a\phi](v)$ 被替换为 $\left[e^{i\phi(v+\varepsilon\delta_a)}-e^{i\phi(v-\varepsilon\delta_a)}\right]e^{-i\phi(v)}$；其中 δ_a 表示局部 a 方向的平移，$\phi(v)-\phi(v_0)$ 本身被替换为 $i^{-1}\int_{c_v}dW(x)W(x)$，$W(x)=\exp(i\phi(x))$ 的 ε 分解黎曼和近似，c_v 是从某个参考点 v_0 到 v 的路径，ϕ 在参考点 v_0 处衰减为零或由经典边界条件固定。或者，也可以使用更一般的函数 $W_\mu(x)=e^{i\mu\phi(x)}$，对固定的 μ 将 $\phi(x)$ 替换为 $\left[W_\mu(x)-W_{-\mu}(x)\right]/(2i\mu)$ [12]。由此产生的算符对 μ 的额外依赖在圈量子宇宙学 (LQC) 文献中广受争议 [1, 9,40](另见第 90 章“圈量子宇宙学: 理论与观测的关系”和 B89 章“圈量子宇宙学: 奇点消解的物理及其启示”)，圈量子宇宙学聚焦于该理论的宇宙学 sector，大量应用了上文为完整理论引入的技术。

The piece C^Y uses all of the above.

项 C^Y 用到了上述全部结论。

An essential feature of the matter contributions is the following: matter can be present only where geometry is excited. This means that the action of the matter contributions is also restricted to those vertices of the generalized SNWF which carries gravitational volume. This physically quite plausible fact which comes out naturally from the formalism plays an important role for the closure of the algebra; see below.

物质贡献的一个核心特征如下: 物质只能存在于几何被激发的区域。这意味着物质贡献的作用也仅局限于广义自旋网波函数中携带引力体积的顶点。这个从形式体系中自然导出的结论在物理上非常合理，对代数的封闭性有重要作用；参见下文。

It is transparent that the complete quantization of the regulated operator has introduced additional quantization ambiguities (e.g., the details of the loop attachment) and the question is how much of that survives upon removal of the regulator ε .

很明显，正则化算符的完整量子化引入了额外的量子化不确定性 (例如，圈附着的细节)，问题在于移除 regulator ε 后，还有多少不确定性会保留下来。

Gauß Constraint and Spatial Diffeomorphism Constraint

高斯约束与空间微分同胚约束

Before we discuss the regulator removal for the Hamiltonian constraint, we first construct the quantum operators corresponding to the remaining constraints as well as their solutions.

在讨论哈密顿约束的调节子消除之前，我们先构造对应剩余约束及其解的量子算符。

As shown in [17, 18, 101] whenever (1) the constraints are linear in momentum and (2) those momenta annihilate the vacuum, which is the case for these constraints, one can obtain unitary operators $U\left(L, \underline{L}, u\right)$ corresponding to their exponentiation $\exp\left(i\left[G\left[L\right]+\underline{G}\left[\underline{L}\right]+D\left[u\right]\right]\right)$ defined densely by

如 [17, 18, 101] 所示，只要 (1) 约束对动量是线性的，且 (2) 这些动量湮灭真空——这正是此处这些约束满足的情况，我们就可以得到对应其指数化 $\exp\left(i\left[G\left[L\right]+\underline{G}\left[\underline{L}\right]+D\left[u\right]\right]\right)$ 的幺正算符 $U\left(L, \underline{L}, u\right)$ ，它按如下方式稠定:

$$U\left(L, \underline{L}, u\right) w\left[F^G, \underline{F}^{YM}, F^F, F^S\right] \Omega$$

$$= w\left[\left(e^X_{L,\underline{L},u} \cdot K\right)\left((0, F), (0, \underline{F}), (0, F^F), (0, F^S)\right)\right] \Omega \tag{19}$$

where $X_{L,\underline{L},u}$ is the Hamiltonian vector field of $G\left[L\right]+\underline{G}\left[\underline{L}\right]+D\left[u\right]$, the Weyl element is given as before by i times the path-ordered exponential of $F\left[A\right]+F\left[A\right]+\ F^F\left[(\eta, v)\right]+F^S\left[\phi\right]$ where F^* denote the smearing functions (real or Lie algebra valued for bosons (see section ”Kinematical Hilbert Space Representations”), Grassmann valued for fermions; see also [97] for more details) of the respective sectors with the smearing dimensions derived above, and K is the momentum coordinate function on the classical phase space. Therefore, (19) is natural, free of any ambiguities, and those constraints close among themselves without anomalies as they should.

其中 $X_{L,\underline{L},u}$ 是 $G[L] + \underline{G}[\underline{L}] + D[u]$ 的哈密顿矢量场，外尔元和之前一样由 i 乘以 $F[A] + F[A] + F^F[(\eta, v)] + F^S[\phi]$ 的路径排序指数给出，F^* 表示弥散函数 (玻色子为实数值或李代数值，见章节 “运动学希尔伯特空间表示”，费米子为格拉斯曼数值；更多细节另见文献 [97])，对应各个扇区，弥散维度已在之前推导得出，K 是经典相空间上的动量坐标函数。因此式 (19) 是自然的，不存在任何歧义，且这些约束符合预期，自闭合无反常。

Note that the generators of one-parameter subgroups exist for the Gauß constraints but not for the spatial diffeomorphism constraints due to the discontinuity of the representation. Thus, while the solution of the Gauß constraints simply restricts the kinematical Hilbert space to its Gauß-invariant subspace (Gauß-invariant SNWF) which just requires harmonic analysis on SU(2) and G [16], the solutions of the spatial diffeomorphism constraints are distributions $\mathcal{D}^*_{\text{diff}}$ that result from averaging over all diffeomorphisms; see, e.g., [11] where also possible Hilbert space structures on those spaces of distributions are discussed.

注意，由于表示不连续，高斯约束存在单参数子群的生成元，而空间微分同胚约束不存在。因此，高斯约束的解仅需将运动学希尔伯特空间限制在其高斯不变子空间 (高斯不变 SNWF)，这只需要对 SU(2) 和 G 做调和分析 [16]，而空间微分同胚约束的解是对所有微分同胚做平均得到的分布 $\mathcal{D}^*_{\text{diff}}$ ；参见例如文献 [11]，其中还讨论了这些分布空间上可能的希尔伯特空间结构。

Regulator Removal from the Hamiltonian Constraint and Operator Topologies

从哈密顿约束中去除正则化项与算子拓扑

The removal of the ε dependence requires the specification of an operator topology, i.e., a notion of convergence. The standard weak or strong operator topologies cannot be used because, e.g., holonomy operators are not weakly or strongly continuous with respect to the path along which they are defined. Apart from the spaces $\mathcal{D}$ and $\mathcal{H}$, the only other natural spaces available at this point are the space $\mathcal{D}^*$ of algebraic distributions over $\mathcal{D}$ (linear functionals without continuity notion) and its subspace $\mathcal{D}^*_{\text{diff}}$ of spatially diffeomorphism-invariant elements discussed in the previous subsection. This suggests to consider a topology that is reminiscent of the so-called weak* operator topology [82]: An open neighborhood base for this topology is given by

去除对 ε 的依赖需要指定一个算子拓扑，即收敛的定义。标准弱算子拓扑或强算子拓扑无法使用，原因在于: 例如，环绕算子相对于其定义路径不满足弱连续或强连续。除空间 $\mathcal{D}$ 和 $\mathcal{H}$ 之外，目前仅有的其他自然空间是定义在 $\mathcal{D}$ 上的代数分布空间 (无连续性概念的线性泛函) $\mathcal{D}^*$ ，以及前一小节讨论过的其空间微分同胚不变元素构成的子空间 $\mathcal{D}^*_{\text{diff}}$ 。这提示我们考虑一种类似所谓弱 * 算子拓扑的拓扑 [82]: 该拓扑的开邻域基由

$$N_\delta(l_1, .., l_m; T_1, .., T_n; O) = \{O'; |l_r[(O' - O)T_s]| < \delta; 1 \le r \le m; 1 \le s \le n\}$$

(20)

where O, O' are in the set of operators with $\mathcal{D}$ as dense invariant domain and $l_r \in \mathcal{D}^*_{\text{diff}}, T_s \in \mathcal{D}$. Early steps toward this topology were stated in [84]; see [92,94,95] for more details.

给出，其中 O, O' 属于以 $\mathcal{D}$ 为稠密不变定义域的算子集合，且满足 $l_r \in \mathcal{D}^*_{\text{diff}}, T_s \in \mathcal{D}$ 。该拓扑的早期研究成果记载于文献 [84]；更多细节参见文献 [92,94,95]。

As the loop attachment is performed in a spatially diffeomorphism-covariant way, one finds that for any smearing function $f, l \in \mathcal{D}^*_{\text{diff}}, T \in \mathcal{D}$,

由于环连接是以空间微分同胚协变的方式完成的，因此可以发现，对任意抹光函数 $f, l \in \mathcal{D}^*_{\text{diff}}, T \in \mathcal{D}$,

$$l\left[\left(C_\varepsilon[f] - C_{\varepsilon_0}[f]\right)T\right] = 0 \tag{21}$$

for any $\varepsilon, \varepsilon_0$ where it is understood that for $\varepsilon > 0$, the prescription of loop attachments and similar finite-size ambiguities is such that it overlaps with the given graph except for an additional ”arc” between two segments of edges adjacent to a given vertex which is attached transversally to their endpoints and that intersects the graph nowhere else and whose braiding is the same for all ε . Using the axiom of choice, we can now pick once and for all some ε_0 and define the regulator-free operator by $C[f] := C_{\varepsilon_0}[f]$.

对任意 $\varepsilon, \varepsilon_0$ 成立，其中约定: 对于 $\varepsilon > 0$ ，环连接及同类有限尺寸模糊性的规定满足: 它与给定图仅在额外一段“弧”处重叠，该弧位于给定顶点相邻的两条边段之间，横向连接到两条边的端点，不与图的其他部分相交，且对所有 ε 其编结都是相同的。利用选择公理，我们现在可以永久选定某个 ε_0 ，并通过 $C[f] := C_{\varepsilon_0}[f]$ 定义无正则化算子。

Commutator Algebra, Closure, and Anomalies

对易子代数、闭合性与反常

We can now check whether the hypersurface deformation algebra $\mathfrak{h}$ is represented without anomalies. Actually, it is not possible to do this for $\mathfrak{h}$ itself, because the operator $D[u]$ does not exist in the chosen representation. We will therefore verify a classically equivalent algebra generated by $\exp(X_u), X_f$ where X_u, X_f are the Hamiltonian vector fields of $D[u], C[f]$, respectively. We have by the general relation between Hamiltonian vector fields and Poisson brackets from (6)

现在我们可以检验超曲面形变代数 $\mathfrak{h}$ 是否无反常表示。事实上，无法直接对 $\mathfrak{h}$ 本身进行检验，因为算子 $D[u]$ 在所选取的表示中不存在。因此我们将验证由 $\exp(X_u), X_f$ 生成的经典等价代数，其中 X_u, X_f 分别是 $D[u], C[f]$ 的哈密顿矢量场。根据 (6) 中给出的哈密顿矢量场与泊松括号的一般关系，我们有

$$e^{X_u} e^{X_v} e^{-X_u} = \exp\left(X_{e^{[u,.]}.v}\right), e^{X_u} X_f e^{-X_u} = X_{e^u.f},$$

$$\left[X_f, X_g\right] = X_{u=-q^{-1}(fdg-gdf)} \tag{22}$$

The last relation cannot be written in the form of e^{X_w} for some, possibly phase space-dependent vector field w , for that one would also have to exponentiate the Hamiltonian constraint which is in fact possible for the $\mathrm{U}(1)^3$ truncation of Euclidian vacuum GR with cosmological constant [17, 18, 101] . Fortunately, in contrast to $D[u]$ itself, the quantity $D\left[q^{-1}(fdg - gdf)\right.$ can be quantized in the chosen representation [96]; hence, we leave (22) as it stands.

最后这个关系无法写成某个依赖相空间的矢量场 w 对应下 e^{X_w} 的形式，若要写成该形式，我们还必须对哈密顿约束做指数化，这对于带宇宙学常数 [17, 18, 101] 的欧几里得真空广义相对论 $\mathrm{U}(1)^3$ 截断而言实际上是可行的。幸运的是，与 $D[u]$ 本身不同，量 $D\left[q^{-1}(fdg - gdf)\right.$ 可以在所选取的表示中量子化 [96]；因此我们保留 (22) 的形式不变。

In the quantum theory, we find

在量子理论中，我们得到

$$U(u)\,U(v)\,U(u)^{-1} = U\left(e^{[u,\cdot]}v\right), U(u)\,C[f]\,U(u)^{-1}T_\gamma$$

$$= C\left[e^{u\cdot f}\right]T_\gamma + \left[\widetilde{U}\left(\varphi_{u,f,\gamma}\right) - 1\right]T_\gamma,$$

$$\Big[C[f], C[g]\,T_\gamma = \sum_{v,v'\in V(\gamma)}\left[f(v)\,g(v') - f(v')\,g(v)\right]\left[\widetilde{U}\left(\varphi_{\gamma,v,v'}\right) - 1\right]C_{\gamma_v,v'}C_{\gamma,v}T_\gamma$$

(23)

where $C_{\gamma,v}$ is the contribution from the vertex v when acting on γ and γ_v is the graph resulting from this action (we are oversimplifying here; see [95] for the details). Here, $\widetilde{U}$ is the extension of U from diffeomorphisms generated by vector fields u to general diffeomorphisms φ , and the diffeomorphisms displayed depend on the structures indicated. Note that the double sum in the second line involves only the vertices of γ and not the new vertices resulting from the first action which is due to the fact that these vertices are co-planar and annihilated by the employed version of the volume operator. This is also the reason for why this works for all matter couplings, the cosmological constant, and both vacuum GR contributions which are all contributing only where geometry is excited.

其中 $C_{\gamma,v}$ 是顶点 v 作用于 γ 时产生的贡献，γ_v 是该作用得到的图 (此处我们做了简化处理；细节参见 [95])。这里，$\widetilde{U}$ 是 U 从矢量场 u 生成的微分同胚到一般微分同胚 φ 的延拓，所示微分同胚依赖于标出的结构。注意第二行的双重求和仅涉及 γ 的顶点，不包含第一次作用产生的新顶点，这是因为这些顶点是共面的，会被所用版本的体积算子湮灭。这也是该构造适用于所有物质耦合、宇宙学常数以及两类真空广义相对论贡献的原因，这些贡献都仅在几何激发处产生贡献。

Thus, the diffeo-diffeo commutator is anomaly-free, the diffeo-Hamiltonian commutator is anomaly-free modulo a term proportional to the exponentiated diffeomorphism constraint $\widetilde{U}(\varphi) - 1$, and the Hamiltonian-Hamiltonian commutator is plain anomalous: it is non-vanishing, and while proportional to linear combinations of the exponentiated diffeo constraint correctly ordered to the outmost left and in that sense closes, i.e., does not lead to new constraints, the ”operators of proportionality” or quantum structure functions are incorrect: as announced, a possible quantization of $D\left[q^{-1}(fdg-gdf)\right]$ is given by (its regulator-free version is obtained in the same topology as for the Hamiltonian constraint and is non-vanishing for discontinuous f,g) [96]

因此, 微分同胚-微分同胚对易子是无反常的, 微分同胚-哈密顿对易子在差一个正比于指数化微分同胚约束 $\widetilde{U}(\varphi)-1$ 的项的意义下无反常，而哈密顿-哈密顿对易子则完全存在反常: 它非零，虽然它正比于指数化微分同胚约束的线性组合，且这些约束正确排序到最左侧，在该意义下对易子是闭合的 (即不会产生新约束), 但“比例算子”即量子结构函数是不正确的: 如前所述, $D\left[q^{-1}(fdg-gdf)\right]$ 的一种可能量子化由下式给出 (其无正则化版本在与哈密顿约束相同的拓扑中得到，且对不连续 f,g 非零)[96]

$$D\left[q^{-1}(fdg-gdf)\right]T_\gamma$$

$$=-\sum_{v\in V(\gamma)}\sum_{e\cap e'=v}\left[f(v)g_e(v)-f_e(v)g(v)\right]\left[\widetilde{U}\left(\varphi_{\gamma,v,e'}\right)-1\right]Q^{e,e'}T_\gamma \tag{24}$$

where $\varphi_{\gamma,v,e'}$ is a diffeomorphism with support in the vicinity of v generated by a vector field which coincides on e' with the tangent vector field of $e', Q^{e,e'}$ is the geometrical operator of the form discussed in section ”Inverse Powers of E and Quantization Ambiguities” which is a quantization of q^{ab} , and $f_e(v) := \lim_{t\to 0} f(e(t))$ is the path-dependent limit of f at v with $e(0)=v$. Formula (24) displays the expected ”quantum structure functions” which differ from those in (23). Note that in the classical theory, one works with continuous, typically even smooth, functions f,g so that the right-hand side of (24) would be zero. The reason for this discrepancy is that in the quantization of the constraints, a central ingredient of the classical theory, namely, the non-degeneracy of the spatial metric, is violated. If one would reinstall non-degeneracy, the discrepancy is likely to disappear. This is further discussed in section ”Quantum Non-degeneracy”.

其中 $\varphi_{\gamma,v,e'}$ 是支撑在 v 邻域内的微分同胚，由一个在 e' 上与 $e', Q^{e,e'}$ 的切向量场重合的向量场生成; $e', Q^{e,e'}$ 是「E 的逆幂与量子化歧义」一节讨论的形式的几何算符，是对 q^{ab} 的量子化，且 $f_e(v) := \lim_{t\to 0} f(e(t))$ 是在 v 处、满足 $e(0)=v$ 条件下 f 依赖路径的极限。公式 (24) 给出了预期的“量子结构函数”，其与 (23) 中的结构函数不同。注意在经典理论中，我们处理的是连续 (通常还是光滑) 的函数 f,g，因此 (24) 的右手边将为零。产生这种差异的原因是，在约束的量子化过程中，经典理论的一个核心要素——空间度量的非退化性——被打破了。如果重新引入非退化性，该差异很可能会消失，这一点在「量子非退化性」一节有进一步讨论。

One reason for the failure of the last relation in (23) to produce (24) is due to the fact that the AL volume operator chosen in the quantization vanishes on the coplanar tri-valent vertices produced by the first action of the Hamiltonian constraint. On the other hand, if it would not (e.g., by using the RS volume), the commutator would not even be a linear combination of exponentiated diffeo constraints. At least in the present form, the Hamiltonian constraint does not enforce solutions to the quantum constraints in addition to the exponentiated diffeo and Hamiltonian constraints, and in that sense, the type of the anomaly is less disastrous than one which leads to downsizing the number of physical degrees of freedom more than in the classical theory. However, improvement of the action of the Hamiltonian constraint on $\mathcal{D}$ must be such that a second action does not vanish at the vertices produced by the first action which is not ruled out to be possible but has not been done yet. See [114] for a toy model where this step could be completed.

(23) 中最后一个关系无法推导出 (24) 的一个原因是: 量子化中选用的 AL 体积算符在哈密顿约束第一次作用产生的共面三价顶点上为零。另一方面，若体积算符不为零 (例如选用 RS 体积)，对易子甚至无法成为指数化微分同胚约束的线性组合。至少在目前的形式下，除了指数化微分同胚约束与哈密顿约束外，哈密顿约束本身不会额外要求量子约束满足新条件；从这个意义上说，这类反常的破坏性比那种会导致物理自由度数量比经典理论缩减更多的反常更小。然而，要改进哈密顿约束在 $\mathcal{D}$ 上的作用，就必须保证第二次作用在第一次作用产生的顶点上不为零——目前并未排除这种可能性，但尚未实现。关于可完成该步骤的玩具模型参见文献 [114]。

To conclude, we distinguish between the notions of closure and non-anomalous: Closure means that a quantum commutator algebra of constraints is a linear combination of those constraints ordered to the left with some structure operators. Non-anomalous means that those structure operators qualify as the quantization of the corresponding classical structure functions. The constructed algebra thus closes but is anomalous. This is better than non-closure, but still a non-anomalous representation is desired.

最后，我们区分闭包性与无反常这两个概念: 闭包性是指约束的量子对易子代数可以表示为左序约束乘以结构算符的线性组合；无反常是指这些结构算符符合对应经典结构函数的量子化要求。因此本文构造的代数满足闭包性但存在反常。这种情况比不闭包更好，但我们仍需要无反常的表示。

On-Shell Closure, Off-Shell Closure, and Habitats

壳闭包、脱壳闭包与生境

By definition, an algebra closes off-shell on some invariant space if that space contains elements not annihilated by all algebra elements. It closes partly (fully) on-shell if the space consists of elements which are annihilated by a subalgebra (the full algebra - in that case, the action of the algebra on the space is trivial).

根据定义，若某个不变空间包含未被所有代数元素零化的元素，则代数在该空间上脱壳闭包。若该空间由被子代数 (整个代数——此时代数在空间上的作用是平凡的) 零化的元素构成，则代数部分 (完全) 壳闭包。

The algebra of the $\widetilde{U}(\varphi) - 1, C[f]$ that we have defined in the previous subsection acts on the dense invariant domain $\mathcal{D}$ which is not annihilated by either the $C[f]$ or the $\widetilde{U}(\varphi) - 1$. It is therefore an off-shell, non-Abelian representation which however is anomalous. This fact is often confused in the literature; see, e.g., [78]: The space $\mathcal{D}^*_{\text{diff}}$ is just used to define a topology w.r.t. which one can remove the regulator ε in $C_\varepsilon[f]$, i.e., to reach the limit $C[f]$. The operator is defined on $\mathcal{D}$ and not on $\mathcal{D}^*_{\text{diff}}$; it cannot possibly be because $C[f]$ is not diffeomorphism invariant so $\mathcal{D}^*_{\text{diff}}$ cannot possibly be an invariant domain. See [113] for more details.

我们在上一小节定义的 $\widetilde{U}(\varphi) - 1, C[f]$ 代数作用在稠密不变定义域 $\mathcal{D}$ 上，该定义域既不被 $C[f]$ 也不被 $\widetilde{U}(\varphi) - 1$ 零化。因此这是一个脱壳的非阿贝尔表示，但它存在反常。这一点在文献中常被混淆；例如见 [78]: 空间 $\mathcal{D}^*_{\text{diff}}$ 仅被用来定义一个拓扑，以便我们在 $C_\varepsilon[f]$ 中移除正则化项 ε，即达到极限 $C[f]$ 。该算子定义在 $\mathcal{D}$ 而非 $\mathcal{D}^*_{\text{diff}}$ 上；它不可能定义在 $\mathcal{D}^*_{\text{diff}}$ ，因为 $C[f]$ 不微分同胚不变，因此 $\mathcal{D}^*_{\text{diff}}$ 不可能是不变定义域。更多细节见 [113]。

Still one can try to define $C[f]$ on a different space as first suggested in [44,47]: It is a subspace of $\mathcal{D}^*$ containing $\mathcal{D}^*_{\text{diff}}$ which however is genuinely larger than it. Therefore, the action of the algebra on that space $\mathcal{D}^*_{\text{VS}}$ of so-called vertex smooth distributions or "habitat" will be off-shell. The space has a basis whose elements l consist of linear combinations of SNWF with the same spin and intertwiner label but where we sum over the diffeomorphism class of the graph label with graph-dependent coefficients. If those coefficients depend continuously on the graph, then the definition $(C'[f]\,l)[T] := \lim_{\varepsilon} l(C_\varepsilon[f]\,T)$ is well defined, and $C'[f]$ leaves $\mathcal{D}^*_{\text{vs}}$ invariant. One shows that their algebra is trivial $[C'[f], C'[g]] = 0$ on this space. Note that there is no contradiction to the previous section because $C[f], C'[f]$ are operators defined on different spaces or, in other words, these are different representations. On both spaces, the algebra closes but with an anomaly, but it does not prevent the existence of non-trivial solutions.

正如文献 [44,47] 最初提出的，我们仍可以尝试在不同空间上定义 $C[f]$: 该空间是 $\mathcal{D}^*$ 的子空间，包含 $\mathcal{D}^*_{\text{diff}}$ ，且真包含 $\mathcal{D}^*_{\text{diff}}$ 。因此，代数在这个被称为顶点光滑分布或“生境”的空间 $\mathcal{D}^*_{\text{VS}}$ 上的作用仍是脱壳的。该空间的一组基的元素 l 是自旋网波函数 (SNWF) 的线性组合，这些 SNWF 具有相同自旋和 intertwiner 标记，但我们对图标记的微分同胚类求和，并带有依赖于图的系数。若这些系数连续依赖于图，则定义 $(C'[f]\,l)[T] := \lim_{\varepsilon} l(C_\varepsilon[f]\,T)$ 是良好定义的，且 $C'[f]$ 保持 $\mathcal{D}^*_{\text{vs}}$ 不变。可以证明，它们的代数在这个空间上是平凡的 $[C'[f], C'[g]] = 0$ 。注意，这与前文并不矛盾，因为 $C[f], C'[f]$ 是定义在不同空间上的算子，换句话说，它们是不同的表示。在两个空间上，代数都是闭包的，只是都带有反常，但这并不妨碍非平凡解的存在。

Note that habitat representations, in contrast to Hilbert space representations of $\mathfrak{h}$, do not come equipped with a Hilbert space structure or other topology. They are thus rather formal objects. If the constraints are not defined on the kinematical HS, rigging methods to solve them and to provide a physical HS structure are not available, and one could in fact have started with formal habitat representations of the CCR without going through the exercise to define HS representations of the CCR and AR (implementation of the AR is impossible outside a Hilbert space context). In fact, how to close the constraint algebra formally using sufficiently differentiable functions has been shown already in [28, 46] for what one could now call the "loop representation habitat." It was defined for Lorentzian self-dual gravity with density weight two, but it applies verbatim to Euclidian gravity. One defined regulated operators by multiplying them with positive powers of ε (multiplicative renormalization) and then taking limits. The resulting algebra is formally closing without anomalies on the chosen habitat. To avoid confusion, note that it is the kinematical HS structure that is needed to define the rigging map, not the operator topology defined above that was merely used to define the continuum limit of the Hamiltonian constraint on the kinematical HS.

请注意，与 $\mathfrak{h}$ 的希尔伯特空间表示不同，生境表示不自带希尔伯特空间结构或其他拓扑。因此它们是相当形式化的对象。如果约束无法在运动学希尔伯特空间上定义，就无法使用索具方法求解约束并提供物理希尔伯特空间结构，事实上我们完全可以从对正则对易关系的形式化生境表示出发，不需要预先构造正则对易关系与自同构群的希尔伯特空间表示 (在希尔伯特空间语境之外无法实现自同构群)。事实上，早在 [28, 46] 中就已经针对如今可称为“圈表示生境”的情形证明了如何利用充分可微函数形式地闭合约束代数。该生境原本是针对密度权重为 2 的洛伦兹自对偶引力定义的，但它可以逐字逐句直接应用于欧几里得引力。研究者通过将算子乘以 ε 的正次幂定义正则化算子 (乘性重整化)，随后取极限。所得代数在选定生境上形式地闭合，没有反常。为避免混淆，需要说明: 定义索积映射需要的是运动学希尔伯特空间结构，而非上文定义的算子拓扑——后者仅用于在运动学希尔伯特空间上定义哈密顿约束的连续极限。

Solutions and Propagation

解与传播

In [95], a general framework was laid out for how to construct generalized (i.e., distributional, elements of $\mathcal{D}^*$) solutions l to all constraints, satisfying $l\left[C\left[f\right]T\right] = l\left[\left(\widetilde{U}\left(\varphi\right) - 1\right)T\right] = 0$ for all $f, \varphi, T \in \mathcal{D}$. Obviously, the space of solutions $\mathcal{D}^*_{\text{phys}}$ is a subspace of $\mathcal{D}^*_{\text{diff}}$; hence, these are certain linear combinations of elements of $\mathcal{D}^*_{\text{diff}}$, and the Hamiltonian constraints impose linear relations on the corresponding coefficients. In particular, one can construct rather simple solutions consisting of linear combinations of diffeo inv. distributions labeled by a small number of diffeomorphism classes of graphs, so that $l\left[C\left[f\right]T_{\gamma,j\iota}\right] = 0$ is not automatically satisfied for a finite number of diffeo classes of γ and by diffeo invariance we can restrict to one representative of each of those classes. The set of relations then involves a countable set of coefficients labeled by spins, intertwiners, and those diffeo classes of graphs.

文献 [95] 提出了一个通用框架，用于构造所有约束的广义 (即分布性的，属于 $\mathcal{D}^*$ 的元素) 解 l ，该解满足对所有 $f, \varphi, T \in \mathcal{D}$ 成立的 $l\left[C\left[f\right]T\right] = l\left[\left(\widetilde{U}\left(\varphi\right) - 1\right)T\right] = 0$ 。显然，解空间 $\mathcal{D}^*_{\text{phys}}$ 是 $\mathcal{D}^*_{\text{diff}}$ 的一个子空间；因此，这些解是 $\mathcal{D}^*_{\text{diff}}$ 中元素的特定线性组合，哈密顿约束对相应系数施加了线性关系。具体而言，可以构造相当简单的解，这类解由少量图微分同胚类标记的微分同胚不变分布的线性组合构成，因此对于有限个 γ 的微分同胚类，$l\left[C\left[f\right]T_{\gamma,j\iota}\right] = 0$ 无法自动满足，且根据微分同胚不变性，我们可以只取每个类的一个代表元。该关系集合涉及由自旋、缠结元和这些图的微分同胚类标记的可数个系数。

The fact that $C\left[f\right]$ acts locally at a vertex suggests that the set of those relations can be solved for each vertex individually, because we can restrict the support of f to one of the vertices of γ . In other words, the sets of relations obtained from the action at different vertices appear to decouple, displaying absence of propagation in the set of solutions [89] which would be physically inacceptable.

$C\left[f\right]$ 在顶点处局部作用这一事实表明，这些关系可以对每个顶点单独求解，因为我们可以将 f 的支集限制在 γ 的某个顶点上。换言之，从不同顶点的作用得到的关系集合似乎是解耦的，这意味着解集中不存在传播效应，而这在物理上是不可接受的 [89]。

In [115], it is argued (see below) that the decoupling of the sets of relations from different vertices does not happen generically. The confusion arises due to the following: In [95], two versions of the Hamiltonian constraint were discussed, the one sketched in sections "Complete Regulated Operator" and "Regulator Removal from the Hamiltonian Constraint and Operator Topologies" and another version which is more similar to the one discussed in [8, 75] which much simplifies the structure of the set of solutions but which indeed leads to absence of propagation because of "unique parentage"; see below. Thus, the conclusion of [89] applies to the second version, the analysis of [115] to the first version.

文献 [115] 指出 (见下文)，来自不同顶点的关系集合解耦的情况一般不会发生。这种混淆来源于以下情况: 文献 [95] 讨论了两种版本的哈密顿约束，一种是“完整正则算子”节和“哈密顿约束的正则化去除与算子拓扑”节中勾勒的版本，另一种版本与 [8, 75] 中讨论的更相似，该版本大大简化了解集合的结构，但确实由于“唯一亲本性”导致不存在传播，详见下文。因此，文献 [89] 的结论适用于第二种版本，文献 [115] 的分析适用于第一种版本。

The basic mechanism displayed in [115] can be sketched as follows (we focus on the geometry contribution):

文献 [115] 展示的基本机制可概述如下 (我们聚焦于几何贡献):

The action of C at a vertex v of a SNWF over γ results in a linear combination of SNWF over graphs γ' differing from γ in one (from C_E^G) or two (from C_L^G) new, so-called extraordinary edges a , between pre-existing ones e, e' intersecting in v such that a intersects e, e' transversally in interior points. Consider a graph γ without extraordinary edges with the property that three of its vertices v_1, v_2, v_3 are such that v_1, v_2 are joined by at least two edges e_1, e_2 and v_2, v_3 by at least two edges f_1, f_2 . In particular, the action of C at v_1 produces a SNWF over a graph γ_1' with one new extraordinary edge a_1 between e_1, e_2 . Likewise, the action at v_3 produces in particular a SNWF over a graph γ_2' with one new edge a_2 between f_1, f_2 . Finally, the action at v_2 produces in particular a SNWF over $\tilde{\gamma}_1, \tilde{\gamma}_2$, respectively, with extraordinary edges $\tilde{a}_1, \tilde{a}_2$ between e_1, e_2 and f_1, f_2 , respectively. Consider now a solution l to all constraints which consists of linear combinations of diffeomorphism orbits of SNWF over graphs γ' differing from γ in one extraordinary edge between e_1, e_2 or f_1, f_2 . Then the equations $l\left[C\,[f]\,T_{\hat{\gamma},j,\iota} = 0\right.$ are automatically satisfied unless $\hat{\gamma}$ is in the diffeo class of γ in which case all equations are satisfied iff they are satisfied for $\hat{\gamma} = \gamma$ (by diffeomorphism invariance). We obtain three sets of equations coming from the action of C at v_1, v_2, v_3 , respectively. However, these equations are not independent of each other: The v_1 and v_3 equations are coupled by the v_2 equations because the graphs $\gamma_I', \tilde{\gamma}_I; i = 1, 2$ are diffeomorphic. If one just solves the v_1, v_3 equations, respectively, and specifies the respective free data, then those two sets of free data are brought into relation by the v_2 equations, which is a sense of propagation. This is the effect of non-unique ”parentage” [73, 122, 123] which already has been there since [95], i.e., modulo diffeomorphisms, a ”parent” SNWF can be in the range, modulo diffeomorphisms, of the action of the Hamiltonian constraint from different vertices acting on different ”child” SNWF.

C 作用于 γ 上空间自旋网泡沫的顶点 v ，会得到 γ' 上空间自旋网泡沫的线性组合，相较于 γ ，这些空间自旋网泡沫新增了 1 条 (来自 C_E^G) 或 2 条 (来自 C_L^G) 所谓的特殊边 a ，加在相交于 v 的原有边 e, e' 之间，且满足 a 与 e, e' 在内部点横截相交。考虑一个不含特殊边的图 γ ，它有三个顶点 v_1, v_2, v_3 ，满足 v_1, v_2 之间至少由两条边 e_1, e_2 连接，v_2, v_3 之间至少由两条边 f_1, f_2 连接。具体而言，C 作用于 v_1 ，会得到图 γ_1' 上的空间自旋网泡沫，在 e_1, e_2 之间新增一条特殊边 a_1 。同理，C 作用于 v_3 ，会得到图 γ_2' 上的空间自旋网泡沫，在 f_1, f_2 之间新增一条特殊边 a_2 。最后，C 作用于 v_2 ，会分别得到 $\tilde{\gamma}_1, \tilde{\gamma}_2$ 上的空间自旋网泡沫，分别在 e_1, e_2 和 f_1, f_2 之间带有特殊边 $\tilde{a}_1, \tilde{a}_2$ 。现在考虑所有约束的一个解 l ，它由 γ' 上空间自旋网泡沫微分同胚轨道的线性组合构成，相较于 γ ，这些图仅在 e_1, e_2 或 f_1, f_2 之间多一条特殊边。此时方程 $l\left[C\,[f]\,T_{\hat{\gamma},j,\iota} = 0\right.$ 自动成立，除非 $\hat{\gamma}$ 属于 γ 的微分同胚类；而在属于 γ 微分同胚类的情况下，根据微分同胚不变性，所有方程成立当且仅当它们对 $\hat{\gamma} = \gamma$ 成立。我们得到分别来自 C 作用于 v_1, v_2, v_3 的三组方程。但这些方程并不独立: v_1 对应的方程和 v_3 对应的方程通过 v_2 对应的方程耦合，因为图 $\gamma_I', \tilde{\gamma}_I; i = 1, 2$ 是微分同胚的。如果我们分别求解 v_1, v_3 对应的方程，给定各自的自由数据，那么这两组自由数据会通过 v_2 对应的方程建立联系，这就是传播的含义。这就是非唯一“亲源”效应 [73, 122, 123]，早在文献 [95] 中就已有研究: 即模去微分同胚后，一个“亲代”空间自旋网泡沫，可以位于不同顶点作用于不同“子代”空间自旋网泡沫的哈密顿约束作用值域内 (模去微分同胚)。

In [115], the simpler $\mathrm{U}(1)^3$ model was considered because as compared to $\mathrm{SU}(2)$ for $\mathrm{U}(1)^3$, one can determine the spectrum of the volume operator in closed form. Even in this case, it is surprisingly difficult to make this mathematically water-tight (existence of solutions, normalizability w.r.t. diffeo inv. scalar product) involving number theory and discrete PDE theory, but there is no reason to believe that the difference between $U(1)^3$ and SU (2) theory leads to absence of propagation.

文献 [115] 研究了更简单的 $U(1)^3$ 模型，原因是与 SU (2) 相比，$U(1)^3$ 可以闭合形式得出体积算子的谱。即便在这种情况下，要让它在数学上无懈可击 (解的存在性、关于微分同胚不变标量积的可归一性) 也出人意料地困难，还需要用到数论和离散偏微分方程理论，但没有理由认为 $U(1)^3$ 和 SU (2) 理论之间的差异会导致传播不存在。

Interim Summary: Anomalies and Ambiguities

中期总结: 反常与歧义

To summarize the developments so far: The Hamiltonian constraint $C[f]$ of [92, 94] is densely defined on a Hilbert space $\mathcal{H}$ carrying a representation of the CCR (or CAR) and AR of geometry and matter. However, (1) it suffers from quantization ambiguities that survive the regulator removal limit $\varepsilon \to 0$ such as those indicated in section "Inverse Powers of E and Quantization Ambiguities", and (2) the constraint algebra generated by it and the spatial diffeomorphism constraint is anomalous in the sense that while the algebra still closes, it closes with the wrong structure operators (quantizations of the structure functions). It could have been worse: The commutators could have resulted not in linear combinations of Hamiltonian and spatial diffeomorphism constraints which would imply that the number of physical quantum degrees of freedom is lower than the number of physical classical degrees of freedom. However, the anomaly indicates that the quantum theory in its present form, while not constraining the wrong number of degrees of freedom, selects the qualitatively wrong physical degrees of freedom.

迄今进展总结如下: [92, 94] 的哈密顿约束 $C[f]$ 在希尔伯特空间 $\mathcal{H}$ 上被稠密定义，该空间承载了几何与物质的 CCR(或 CAR) 和 AR 的一个表示。但存在两个问题:(1) 它存在去除正则化极限 $\varepsilon \to 0$ 后仍保留的量子化歧义，例如“E 的逆幂与量子化歧义”一节中指出的这类歧义；(2) 由它和空间微分同胚约束生成的约束代数存在反常: 虽然该代数仍然封闭，但却是以错误的结构算符 (结构函数的量子化结果) 封闭。情况本可能更糟: 对易子本可能无法得到哈密顿约束与空间微分同胚约束的线性组合，这意味着物理量子自由度的数量会少于物理经典自由度的数量。但该反常表明，目前形式的量子理论虽然没有限制出错误数量的自由度，却选择了定性上错误的物理自由度。

Clearly, the ambiguities and the anomaly must be adequately dealt with. These two problems are very likely linked to each other as one would expect that anomaly avoidance decreases the amount of quantization ambiguity. The developments that will be described below seek to avoid the anomaly in different ways. In the master constraint approach [48, 49, 51, 52, 109] , one avoids the algebra $\mathfrak{h}$ altogether by using a classically equivalent single constraint. In the electric shift approach [73,122,123], one uses different density weights and habitats to construct a representation of $\mathfrak{h}$ in the sense of [44]. In the reduced phase space approach [39, 50, 55, 57, 68] , to which we devote a section of its own, the algebra $\mathfrak{h}$ is dealt with classically so that quantum anomalies cannot possibly arise. However, all three approaches still suffer from quantization ambiguities. Therefore, e.g., (Hamiltonian) non-perturbative renormalization [99] must be used as an additional step to remove the quantization ambiguities.

显然，必须妥善处理歧义和反常。这两个问题很可能相互关联，因为人们会预期避免反常会减少量子化歧义的数量。下文将要描述的研究进展尝试通过不同方式避免反常。在主约束方案 [48, 49, 51, 52, 109] 中，人们通过使用一个经典等价的单一约束，完全避开了代数 $\mathfrak{h}$。在电移位方案 [73,122,123] 中，人们使用不同的密度权重和生存空间，按照文献 [44] 的思路构造 $\mathfrak{h}$ 的一个表示。在约化相空间方案 [39, 50, 55, 57, 68] 中 (我们会为它单独设一节讨论)，代数 $\mathfrak{h}$ 在经典层面就得到处理，因此量子反常不可能出现。但这三种方案仍然都存在量子化歧义，因此必须额外采用例如 (哈密顿) 非微扰重整化 [99] 来消除量子化歧义。

Assuming that the anomalies can be properly removed, compared to the perturbative QFT (or effective FT) approach to QG, the progress of LQG to date can be stated as follows:

假设反常可以被妥善消除，相较于量子引力 (QG) 的微扰量子场论 (或有效场论) 方法，LQG 迄今取得的进展可总结如下:

1. In the EFT approach, there are short-distance (UV) infinities. In LQG, there are no UV infinities due to spatial diffeomorphism invariance.

1. 在有效场论方法中存在短距离 (紫外) 无穷大。在 LQG 中，由于空间微分同胚不变性，不存在紫外无穷大。

2. In the EFT approach, even after taming the UV infinities using perturbative renormalization, there is a perturbation series to be summed with little control over the radius of convergence. In LQG, there is no series to be performed because the approach is background independent and non-perturbative from the outset.

2. 在有效场论方法中，即便用微扰重整化处理了紫外无穷大，仍需要对微扰级数求和，且人们对收敛半径几乎没有控制。在 LQG 中不存在需要计算的级数，因为该方法从一开始就是背景独立且非微扰的。

3. In the EFT approach, perturbative renormalization of the UV infinities requires adding new counterterms order by order making the theory non-predictive. In LQG, there are no UV infinities and thus no counterterms required. However, there are quantization ambiguities which also make the theory non-predictive so far. This stresses again the necessity to use non-perturbative renormalization in LQG.

3. 在有效场论方法中，对紫外无穷大做微扰重整化要求逐阶添加新的抵消项，导致理论失去预言能力。在 LQG 中不存在紫外无穷大，因此不需要抵消项。但依然存在量子化歧义，这导致该理论目前同样不具备预言能力，这再次强调了在 LQG 中使用非微扰重整化的必要性。

(Extended) Master Constraint

(推广) 主约束

The idea of the master constraint approach is quite simple: Given a set of classical first-class constraints C_I possibly with structure functions rather than structure constants, consider instead the master constraint

主约束方法的思路十分简单: 给定一组具有结构函数而非结构常数的经典第一类约束 C_I ，我们转而考虑主约束

$$\mathrm{M}=\frac{1}{2}\sum_I C_I^*\omega_I C_J \tag{25}$$

(in the classical theory, the constraints are real-valued $C_I = C_I^*$) where $\omega_I > 0$ are positive (weight) numbers. Then

(经典理论中，约束是实值的 $C_I = C_I^*$)，其中 $\omega_I > 0$ 为正 (权重) 数，由此可得

$$C_I = 0\forall I \Leftrightarrow \mathrm{M}=0,\ \{C_I, O\}_{\mathrm{M}=0}\forall I \Leftrightarrow \{\{\mathrm{M}, O\}, O\}_{\mathrm{M}=0}=0 \tag{26}$$

Thus, M encodes not only the constraint surface but also the observables (reduced phase space). Hence, we consider constructing the single operator M rather than all the C_I . Since the quantization of C_I is however an integral part of quantizing M , the question arises how M encodes the anomaly and what influence the choice of ω_I has. To see this, suppose that zero is in the point spectrum of M and that $M\psi = 0$. Then $<\psi, \mathrm{M}\psi>=\sum_I \omega_I \left\|C_I\psi\right\|^2 = 0$; thus, $C_I\psi = 0\forall I$ and vice versa. Thus, the presence of an anomaly in the C_I will be encoded in the spectrum of M . Removing the anomaly can then be considered as the problem to quantize M such that it has a kernel at all or to minimize the lower bound of the spectrum of M. This proposal has been studied in various models [38] where also the case of zero being part of the continuous spectrum or mixed cases was treated.

因此，M 不仅编码了约束面，还编码了可观测量 (约化相空间)。据此，我们选择构造单个算子 M 而非所有 C_I 。但由于 C_I 的量子化本身就是 M 量子化的核心部分，由此产生了以下问题: M 如何编码反常，以及 ω_I 的选择会带来何种影响。为此，假设零属于 M 的点谱，且满足 $M\psi = 0$ ，此时有 $<\psi, \mathrm{M}\psi>=\sum_I \omega_I \left\|C_I\psi\right\|^2 = 0$ ；因此可得 $C_I\psi = 0\forall I$ ，反之亦然。由此可知，C_I 中反常的存在会被编码在 M 的谱中。消除反常就等价于这样对 M 量子化: 让 M 整体存在核，或是最小化 M 谱的下界。该方案已在多种模型 [38] 中得到研究，零属于连续谱以及混合谱的情况也已在这些研究中得到处理。

For LQG, the following concrete expression was quantized [48,49,51,52,109]:

对于圈量子引力 (LQG)，以下具体表达式已完成量子化 [48,49,51,52,109]:

$$\mathrm{M}=\frac{1}{2}\int_\sigma d^3x\left\{\left[\frac{C}{Q^{1/2}}\right]^*\left[\frac{C}{Q^{1/2}}\right]+\delta^{jk}\left[\frac{E_j^aD_a}{Q^{3/2}}\right]^*\left[\frac{E_k^bD_a}{Q^{3/2}}\right]\right\} \tag{27}$$

The selection of the weight functions is motivated by spatial diffeomorphism invariance. Note that not only the Hamiltonian constraint but also the spatial diffeomorphism constraint is encoded in M (extended master constraint). This has three reasons: First, in contrast to D_a itself, the function $D_j := E_j^a D_a/Q$ or its "square" $q^{ab}D_aD_b/Q = \delta^{jk}D_jD_k/Q$ can be quantized [96]. This kind of operator is also considered in the electric shift approach. Second, using the constraints C and D_j instead of C, D_a makes their algebraic structure more alike. Finally, using the constraints C, D_j , we obtain a closed algebra with the advantage that in the quantum theory, we can use D_j itself rather than its exponentiation. There is also a version that just involves C and which defines an operator on the Hilbert space extension of $\mathcal{D}^*_{\text{diff}}$ (non-extended master constraint).

权重函数的选择是由空间微分同胚不变性推动的。注意，不仅哈密顿约束，空间微分同胚约束也被编码在 M 中 (推广主约束)。这有三点原因: 第一，和 D_a 本身不同，函数 $D_j := E^a_j D_a/Q$ 或其“平方” $q^{ab}D_aD_b/Q = \delta^{jk}D_jD_k/Q$ 可以被量子化 [96]。电位移方法中也考虑了这类算子。第二，使用约束 C 和 D_j 而非 C, D_a，能让它们的代数结构更相似。最后，使用约束 C, D_j，我们可以得到闭合代数，其优势在于量子理论中我们可以直接使用 D_j 本身，无需对它做指数化。也存在仅包含 C 的版本，它在 $\mathcal{D}^*_{\text{diff}}$ 的希尔伯特空间延拓上定义了一个算子 (非推广主约束)。

As the operator (27) is spatially diffeomorphism invariant, it must act on the kinematical Hilbert space in a non-graph-changing way. This can be done as follows: Given a vertex v of a graph γ with two edges e, e' outgoing from v, a loop α in γ based on v, e, e' is called minimal iff it starts from v along e and ends at v along $(e')^{-1}$ and there is no other such loop with fewer edges traversed. The operator M uses such minimal loops instead of the extraordinary edges of $C[f]$ and projection operators making sure that the image of M on a SNWF over γ is a linear combination of SNWF over γ and not a smaller graph. As M is graph preserving, its classical limit can be studied using the coherent state technology of [103, 104, 107, 112] and has been confirmed to converge to the classical expression plus quantum corrections for sufficiently large and fine graphs which justifies the above definition.

由于算符 (27) 具有空间微分同胚不变性，它必定以不改变图的方式作用于运动学希尔伯特空间。具体操作如下: 给定图 γ 的一个顶点 v，有两条边 e, e' 从 v 出发，则基于 v, e, e'、位于 γ 内的圈 α 是极小圈当且仅当它从 v 出发沿 e 行走，最终沿 $(e')^{-1}$ 回到 v，且不存在其他经过更少边的同类圈。算符 M 使用这类极小圈，而非 $C[f]$ 的特殊边与投影算符，以此保证 M 作用在 γ 上的自旋网波函数 (SNWF) 的像仍是 γ 上自旋网波函数的线性组合，而非更小图上的线性组合。由于 M 是保图的，可利用 [103, 104, 107, 112] 的相干态技术研究其经典极限；已有研究证实，对于足够大且足够精细的图，该算符收敛到经典表达式加上量子修正，这证明上述定义是合理的。

Algebraic Quantum Gravity (AQG)

代数量子引力 (AQG)

As mentioned in the previous subsection, good semiclassical properties for M are obtained for coherent states based on a single sufficiently large and fine graph. The non-separable kinematical Hilbert space is spanned by SNWF over finite graphs, and there cannot be a semiclassical state on a single graph that controls the fluctuations of all quantum degrees of freedom, not even if we allow also graphs with a countably infinite number of edges (infinite tensor product (ITP) extension [105]). On the other hand, it is clear that the kinematical Hilbert space is in some sense unnecessarily large: In the classical theory, e.g., a countable number of holonomies and fluxes would suffice to separate the points of the classical phase space. This has motivated the algebraic quantum gravity (AQG) viewpoint: One considers a countable number of quantum degrees of freedom that define an abstract *-algebra $\mathfrak{A}$. The information of how to think of these algebra elements in terms of holonomies and fluxes along embedded paths and surfaces is part of the definition of a semiclassical state. In this way, the universe becomes a single abstract and infinite lattice λ (i.e., only information which vertices are linked by which edges is provided).

如前一小节所述，对于基于单个足够大且精细的图构造的相干态，M 可获得良好的半经典性质。不可分离的运动学希尔伯特空间由有限图上的 SNWF 张成，且不存在单图上的半经典态可以控制所有量子自由度的涨落，即便我们允许图包含可数无穷多条边 (无穷张量积 (ITP) 推广 [105]) 也不行。另一方面，运动学希尔伯特空间在某种意义上显然过大: 例如在经典理论中，可数个全纯和通量就足以区分经典相空间中的不同点。这催生了代数量子引力 (AQG) 的观点: 我们考虑可数个量子自由度，它们定义了一个抽象 * 代数 $\mathfrak{A}$ 。如何根据嵌入路径和曲面上的全纯与通量来理解这些代数元，是半经典态定义的一部分。通过这种方式，宇宙成为单个抽象无穷格 λ (即仅保留顶点由哪条边连接的信息)。

The corresponding Hilbert space representation of $\mathfrak{A}$ is formally the same as in LQG. However, now the spatial diffeomorphism constraint is no longer considered an extra structure but encoded in M. Spatial diffeomorphisms no longer act on λ but just on $\mathfrak{A}$. The reason for why M had to be graph preserving on the LQG Hilbert space now no longer applies, and M spreads its action unlimitedly over λ . The semiclassical analysis of the previous section still applies unchanged.

$\mathfrak{A}$ 对应的希尔伯特空间表示在形式上与 LQG 中的相同。但此时空间微分同胚约束不再被视作额外结构，而是被编码在 M 中。空间微分同胚不再作用于 λ ，仅作用于 $\mathfrak{A}$ 。原先要求 M 在 LQG 希尔伯特空间上保持图结构的理由已不再成立，M 可将其作用无限制扩散到整个 λ 上。前一节的半经典分析无需修改仍可适用。

The AQG viewpoint rests on the selection of the abstract lattice λ . Using the huge freedom for how to embed λ and since one can choose to leave some of its edges non-excited, one can in fact accommodate an uncountably infinite number of what one would call embedded SNWF. Therefore, AQG looks almost like a continuum theory which can be restricted to finite resolution as one desires and thus takes a large step toward non-perturbative renormalization. In fact, in AQG quantization, ambiguities prevail even if one manages to minimize the spectral gap of M , and one would prefer to get rid of the λ dependence.

AQG 的观点基于对抽象格 λ 的选取。由于嵌入 λ 的方式拥有极大自由度，且可选择令部分边保持非激发态，实际上可以容纳不可数无穷多的所谓嵌入自旋网顶点基 (SNWF)。因此，AQG 看起来近乎一种连续统理论，还可按需求限制为有限分辨率，因此向非微扰重整化迈出了一大步。实际上，在 AQG 量子化中，即便成功缩小了 M 的谱隙，仍存在大量歧义，人们更希望消除对 λ 的依赖。

Renormalization

重正化

This then gives direct motivation to enter non-perturbative (Hamiltonian) renormalization (HR) of LQG [99]. It applies simultaneously to the master constraint M of Dirac quantization and to the physical Hamiltonian $\mathbf{H}$ of reduced phase space quantization. As $\mathbf{H}$ is subject of the next section, we focus here exemplarily on $\mathbf{M}$.

这就直接推动了对圈量子引力 (LQG) 开展非微扰 (哈密顿) 重正化 (HR) 研究 [99]。该方法可同时应用于狄拉克量子化的主约束 M 和约化相空间量子化的物理哈密顿量 $\mathbf{H}$ 。由于 $\mathbf{H}$ 是下一节的讨论主题，我们在此示例性地聚焦于 $\mathbf{M}$ 。

The concrete proposal of [99] is motivated by constructive QFT [61], but of course, there were many earlier and related works; see [99] and references therein. In constructive QFT (CQFT), one works with a family of theories labeled by both IR and UV cut-off R, ε , respectively. Then one first removes $\varepsilon \to 0$ using the renormalization flow and after that takes the thermodynamic limit $R \to \infty$. Consider the 3-torus $\sigma = T^3$ of radius R and fields on σ with periodic boundary conditions. We consider cubic lattices on σ with $\varepsilon^{-1} \in \mathbb{N}$ vertices in each direction. We define a partial order on the set $\mathcal{E}$ of resolutions ε by $\varepsilon' \le \varepsilon$ iff $\frac{\varepsilon}{\varepsilon'} \in \mathbb{N}$ is integral so that the coarser lattice is a sublattice of the former. The set $\mathcal{E}$ is also directed this way. Using some technology from wavelet theory [29,102] (multiresolution analysis (MRA)), one constructs a one-particle Hilbert space V of smearing functions of the fields and finite resolution subspaces V_ε with $V_\varepsilon \subset V_{\varepsilon'}, \varepsilon' \le \varepsilon$ thereof together with embeddings $I_\varepsilon : L_\varepsilon \to V_\varepsilon \subset V$ where L_ε is an ℓ_2 space and I_ε is an isometry. This is all one needs to define the coarse graining map $I_{\varepsilon\varepsilon'} = I_{\varepsilon'}^\dagger I_\varepsilon : L_\varepsilon \to L_{\varepsilon'}$ and the projection $p_\varepsilon = I_\varepsilon I_\varepsilon^\dagger : V \to V$.

文献 [99] 的具体方案受构造量子场论 [61] 启发，当然，此前已有大量相关研究工作；参见 [99] 及其中的参考文献。在构造量子场论 (CQFT) 中，我们研究的是同时由红外和紫外截断 R, ε 分别标记的一族理论。随后先利用重正化流移除 $\varepsilon \to 0$ ，之后再取热力学极限 $R \to \infty$ 。考虑半径为 R 的 3 环面 $\sigma = T^3$ ，以及定义在 σ 上满足周期性边界条件的场。我们在 σ 上构造立方晶格，每个方向有 $\varepsilon^{-1} \in \mathbb{N}$ 个顶点。我们对分辨率 ε 的集合 $\mathcal{E}$ 定义偏序: 当且仅当 $\frac{\varepsilon}{\varepsilon'} \in \mathbb{N}$ 为整数时满足 $\varepsilon' \le \varepsilon$ ，此时更粗的晶格是原晶格的子晶格。通过这种方式，集合 $\mathcal{E}$ 也成为有向集。借助小波理论 [29,102] 的多分辨率分析 (MRA) 技术，可以构造场涂抹函数的单粒子希尔伯特空间 V ，以及有限分辨率子空间 V_ε (其中 $V_\varepsilon \subset V_{\varepsilon'}, \varepsilon' \le \varepsilon$) 和嵌入映射 $I_\varepsilon : L_\varepsilon \to V_\varepsilon \subset V$ ，其中 L_ε 是 ℓ_2 空间，I_ε 是等距映射。这些就是定义粗粒化映射 $I_{\varepsilon\varepsilon'} = I_{\varepsilon'}^\dagger I_\varepsilon : L_\varepsilon \to L_{\varepsilon'}$ 和投影 $p_\varepsilon = I_\varepsilon I_\varepsilon^\dagger : V \to V$ 所需的全部内容。

One considers the discretized fields $\left(\phi_\varepsilon := I_\varepsilon^\dagger \phi, \pi_\varepsilon := I_\varepsilon^\dagger \pi\right)$ where (ϕ, π) is a collective notation for the continuum fields and discretized Weyl elements $w_\varepsilon [F_\varepsilon] = w [I_\varepsilon F_\varepsilon]$ with $w[F]$ the continuum Weyl element which is a functional of $< F, \phi >_V$. As an initial discretization of M at resolution ε , we pick $\mathrm{M}_\varepsilon^{(0)} [\phi_\varepsilon, \pi_\varepsilon] := M [p_\varepsilon \phi, p_\varepsilon \pi]$. The discretized fields are canonically conjugate if the continuum ones are, and since in the presence of both cut-offs the number of degrees of freedom is finite, there is a unique Hilbert space representation $(\mathcal{H}_\varepsilon, \rho_\varepsilon)$ of the discretized CCR and AR (Stone-von Neumann theorem). We consider the ground state $\Omega_\varepsilon^{(0)} \in \mathcal{H}_\varepsilon$ and now construct a renormalization flow or sequence of families of pairs $n \mapsto \left(\Omega_\varepsilon^{(n)}, M_\varepsilon^{(n)}\right)_{\varepsilon \in \mathcal{E}}$ as follows: Pick a function $\kappa : \mathcal{E} \to \mathcal{E}; \kappa(\varepsilon) < \varepsilon$ (often $\kappa(\varepsilon) = \varepsilon/2$) and define $\Omega_\varepsilon^{(n+1)}$ such that

我们研究离散场 $\left(\phi_\varepsilon := I_\varepsilon^\dagger \phi, \pi_\varepsilon := I_\varepsilon^\dagger \pi\right.$ ，其中 (ϕ, π) 是连续场与离散外尔元 $w_\varepsilon [F_\varepsilon] = w [I_\varepsilon F_\varepsilon]$ 的统称，$w[F]$ 为连续外尔元，是 $< F, \phi >_V$ 的泛函。作为分辨率 ε 下 M 的初始离散化，我们选取 $\mathrm{M}_\varepsilon^{(0)} [\phi_\varepsilon, \pi_\varepsilon] := M [p_\varepsilon \phi, p_\varepsilon \pi]$ 。若连续场是正则共轭的，则离散场也为正则共轭；且由于同时存在两种截断时自由度数目有限，离散对易关系 (CCR) 和酉关系 (AR) 存在唯一的希尔伯特空间表示 $(\mathcal{H}_\varepsilon, \rho_\varepsilon)$ (由斯通-冯·诺依曼定理保证)。我们考虑基态 $\Omega_\varepsilon^{(0)} \in \mathcal{H}_\varepsilon$ ，然后按如下方式构造重整化流，即对偶对族序列 $n \mapsto \left(\Omega_\varepsilon^{(n)}, M_\varepsilon^{(n)}\right)_{\varepsilon \in \mathcal{E}}$: 选取函数 $\kappa : \mathcal{E} \to \mathcal{E}; \kappa(\varepsilon) < \varepsilon$ (通常取 $\kappa(\varepsilon) = \varepsilon/2$)，定义 $\Omega_\varepsilon^{(n+1)}$ 使得

$$J_{\varepsilon\varepsilon'} w_\varepsilon [F_\varepsilon] \Omega_\varepsilon^{(n+1)} := w_{\varepsilon'} [I_{\varepsilon\varepsilon'} F_\varepsilon] \Omega_{\varepsilon'}^{(n)} \tag{28}$$

is an isometry for $\varepsilon' = \kappa(\varepsilon)$ and set for $\varepsilon' = \kappa(\varepsilon)$

对 $\varepsilon' = \kappa(\varepsilon)$ 是一个等距映射，并对 $\varepsilon' = \kappa(\varepsilon)$ 设定

$$\mathrm{M}_{\varepsilon}^{(n+1)} := J_{\varepsilon\varepsilon'}^{\dagger} \mathrm{M}_{\varepsilon'}^{(n)} J_{\varepsilon\varepsilon'} \tag{29}$$

At a fixed point of the flow (28),(29), we obtain the continuum Hilbert space $\mathcal{H}$ as the inductive limit of the $\mathcal{H}_{\varepsilon}$, i.e., there exist isometries $J_{\varepsilon} : \mathcal{H}_{\varepsilon} \to \mathcal{H}$ with $J_{\varepsilon\varepsilon'} = J_{\varepsilon'}^{\dagger} J_{\varepsilon}$ for $\varepsilon' = \kappa(\varepsilon)$ and a continuum quadratic form M densely defined on the subspaces $J_{\varepsilon}\mathcal{H}_{\varepsilon}$ such that $J_{\varepsilon}^{\dagger} \mathrm{M} J_{\varepsilon} = \mathrm{M}_{\varepsilon}$ (”blocked from the continuum”).

在流 (28),(29) 的不动点处，我们得到连续希尔伯特空间 $\mathcal{H}$ 为 $\mathcal{H}_{\varepsilon}$ 的归纳极限，即存在等距映射 $J_{\varepsilon} : \mathcal{H}_{\varepsilon} \to \mathcal{H}$ 满足对 $\varepsilon' = \kappa(\varepsilon)$ 有 $J_{\varepsilon\varepsilon'} = J_{\varepsilon'}^{\dagger} J_{\varepsilon}$，且在子空间 $J_{\varepsilon}\mathcal{H}_{\varepsilon}$ 上稠定义了一个连续二次型 M 满足 $J_{\varepsilon}^{\dagger} \mathrm{M} J_{\varepsilon} = \mathrm{M}_{\varepsilon}$ (“从连续截断得到”)。

Although difficult to prove in general, one relies on universality and hopes that the inductive limit and quadratic form construction do not depend on the MRA and the map κ which is true in the examples considered so far. The renormalization flow has the tendency to reduce the number of ambiguities (couplings) to the ”relevant and marginal” ones which yields a predictive theory if these are finite in number which is the whole point of renormalization. There is no guarantee that the positive quadratic form M extends to an operator (if it does, one can take its Friedrichs s.a. extension). This program is still in its infancy and has been applied only to solvable QFT models so far.

尽管总体上难以证明，但人们依赖普适性，并期望归纳极限和二次型构造不依赖于 MRA 与映射 κ——在目前研究的例子中这一结论成立。重整化流倾向于将歧义 (耦合常数) 的数量缩减到仅保留“相关和边缘”耦合；若这些耦合数量有限，就能得到一个可预言的理论，这正是重整化的核心意义。无法保证正二次型 M 总能延拓为一个算子 (若可以延拓，则可取其弗里德里希自伴延拓)。该方案目前仍处于起步阶段，仅应用于可解量子场论模型。

One can also apply the same flow equations to the constraints themselves rather than M in which case Ω_{ε} is in general just a cyclic vector rather than a vacuum. The finite resolution algebra blocked from the continuum must always be anomalous even if the continuum algebra is non-anomalous; thus, the ”finite resolution anomaly,” which should better be called a ”discretization artifact,” is physically correct. See [106] for an exactly solvable model and a technical explanation for the phenomenon.

也可以将同一流方程应用于约束本身而非 M，这种情况下 Ω_{ε} 通常只是一个循环向量而非真空。即便连续谱代数没有反常，从连续谱截断得到的有限分辨率代数也必然存在反常；因此所谓“有限分辨率反常”，更准确应当称为“离散化人为误差”，它在物理上是合理的。关于该现象的精确可解模型和技术解释参见文献 [106]。

Electric Shift Approach

电位移方法

The electric shift approach has a longer history starting with first ideas tested in parametrized field theory [70, 71, 74, 119] and then was generalized and improved in the $U(1)^3$ truncation of Euclidian vacuum GR [88, 116, 120] culminating recently in the treatment of full SU(2) Euclidian vacuum GR [73, 122, 123]. Common to these works is that one seeks a more geometrically motivated action of the Hamiltonian constraint than chosen in [92, 94] which is more inspired by lattice gauge theory. In particular, one exploits the closeness of the generator of spatial diffeomorphisms and the Hamiltonian constraint which holds only for Euclidian vacuum GR, i.e., only for the contribution C_E^G to the full Hamiltonian constraint. Therefore, the following exposition is strictly restricted to Euclidian vacuum GR. While there is possibly a chance to extend this to the Lorentzian regime using the quantum Wick transform [93, 121] (see however the reservations spelled out in [73,122,123]), the cosmological constant and matter contributions (and perhaps also the Lorentzian geometry contribution) are excluded. In fact, these contributions are excluded for two reasons: first, because the close relation between diffeo constraint and Hamiltonian constraint does not extend to these contributions and second, because of the necessity to change the density weight.

电位移方法历史更长，其最初构想在参数化场论 [70, 71, 74, 119] 中得到检验，随后在欧几里得真空广义相对论 [88, 116, 120] 的 $U(1)^3$ 截断中得到推广与改进，近期最终形成了完整 SU(2) 欧几里得真空广义相对论的处理方案 [73, 122, 123]。这些工作的共同点是，相较于 [92, 94] 中更受格点规范理论启发的选取方式，本文寻求一种更具几何动机的哈密顿约束作用。具体而言，我们利用了空间微分同胚生成元与哈密顿约束的紧密关联，这种关联仅对欧几里得真空广义相对论成立，即仅适用于完整哈密顿约束中的 C_E^G 贡献项。因此，下文阐述严格限定于欧几里得真空广义相对论。虽然存在可能利用量子威克变换将其扩展到洛伦兹区域 [93, 121](但参见 [73,122,123] 中给出的保留意见)，但宇宙学常数和物质贡献 (或许还有洛伦兹几何贡献) 都被排除在外。实际上，排除这些贡献有两点原因: 第一，微分同胚约束与哈密顿约束之间的紧密关系无法延伸到这些贡献；第二，需要改变密度权重。

Namely, the electric shift approach considers a representation of $\mathfrak{h}$ on a certain habitat, i.e., a certain subspace of the algebraic dual $\mathcal{D}^*$; see section "On-Shell Closure, Off-Shell Closure, and Habitats" and [44]. To avoid the anomalous, Abelian character of the (dual) algebra discovered in [44], in [73, 122, 123], one takes a drastic step: One changes the density weight away from unity. This means that a regulator limit on the kinematical Hilbert space or rather the dense and invariant domain $\mathcal{D}$ in the sense of sections "Smearing Dimensions and Density Weights" and "Regulator Removal from the Hamiltonian Constraint and Operator Topologies" does not exist. This by itself may be argued not to be problematic because one could simply not care about a representation of $\mathfrak{h}$ on $\mathcal{D}$ and is satisfied with a dual representation on a suitable habitat (see however the reservations at the end of section "On-Shell Closure, Off-Shell Closure, and Habitats"). However, even to define the cosmological constant term with the modified density weight on that habitat is not possible [73, 91, 122, 123] ; see section "Smearing Dimensions and Density Weights". Work is in progress to return to density weight unity [73, 122, 123] .

具体而言，电位移方法考虑 $\mathfrak{h}$ 在某一住境 (即代数对偶 $\mathcal{D}^*$ 的某个子空间) 上的表示；参见“在壳闭合、离壳闭合与住境”一节和文献 [44]。为了避免 [44] 中发现的 (对偶) 代数具有反常阿贝尔特征，文献 [73, 122, 123] 采取了一项大胆举措: 将密度权重改为非单位值。这意味着，在运动学希尔伯特空间，更准确说是“涂抹维度与密度权重”和“从哈密顿约束去除正则化与算子拓扑”两节意义下的稠密不变定义域 $\mathcal{D}$ 上，不存在正则化极限。这一点本身可以说不成问题，因为我们大可以不关心 $\mathfrak{h}$ 在 $\mathcal{D}$ 上的表示，只要满足于合适住境上的对偶表示即可 (但参见“在壳闭合、离壳闭合与住境”一节末尾的保留意见)。然而，即便是在该住境上定义带修改密度权重的宇宙学常数项也不可能 [73, 91, 122, 123]；参见“涂抹维度与密度权重”一节。目前已有工作在推进回归单位密度权重 [73, 122, 123] 。

The reason for calling this the electric shift approach is the following: Recall that the density weight w Hamiltonian and spatial diffeomorphism constraint for Euclidian vacuum GR are, respectively, given by (modulo the SU(2) Gauß constraint)

该方法被命名为电位移方法的原因如下: 回想欧几里得真空广义相对论的密度权重 w 哈密顿约束与空间微分同胚约束，分别由下式给出 (模去 SU(2) 高斯约束)

$$C_E^G[f] = \int_\sigma d^3x \left[\frac{fE_j^a}{Q^{2-w}}\right]\left[\varepsilon_{jkl}F_{ab}^k E_l^b\right], D_a^G[u] = \int_\sigma d^3x u^a \left[F_{ab}^k E_k^b\right]$$

(30)

where we have split the factors suggestively. Then $C_E^G[f]$ looks almost as a diffeomorphism with field-dependent (electric) vector field (shift)

其中我们已经做了提示性分解。此时 $C_E^G[f]$ 几乎就是一个带场依赖 (电) 矢量场 (位移) 的微分同胚

$$u_j^a(f) = \frac{fE_j^a}{Q^{2-w}} \tag{31}$$

proportional to f (the lapse function). The rough idea is then to quantize u_j^a independently from the rest of the constraint and to treat the rest along the lines of the spatial diffeomorphism constraint. In fact, in the $U(1)^3$ model, the electric shift operator is diagonal in the analog of the SNWF basis so that $u_j^a(f)$ becomes basically a vector field depending on the SNWF labels and one can then truly proceed in analogy to the quantization of the spatial diffeomorphism constraint. In the SU(2) theory, this is less trivial, but the basic idea is the same.

与 f (lapse 函数，即时移函数) 成正比。大致思路是，将 u_j^a 与约束的其余部分独立量子化，再按照空间微分同胚约束的思路处理其余部分。实际上，在 $U(1)^3$ 模型中，电位移算子在自旋网波函数 (SNWF) 基的类比下是对角的，因此 $u_j^a(f)$ 基本上成为一个依赖于自旋网波函数标签的矢量场，之后我们就完全可以仿照空间微分同胚约束的量子化过程处理。在 SU(2) 理论中，这没那么平凡，但基本思路一致。

The constraint algebra of the Euclidian vacuum Hamiltonian constraints at density weight w is given by and again factorized suggestively

密度权重 w 下欧几里得真空哈密顿约束的约束代数由下式给出，且同样做了提示性分解

$$K_E^G(f,g) := \{C_E^G[f], C_E^G[g]\} = -\int_\sigma d^3x \left[F_{ab}^k E_k^c\right]\left[\frac{E_j^a E_l^b \delta^{jl}}{Q^{4-2w}}\omega_b(f,g)\right],$$

$$\omega_b = fg_{,b} - gf_{,b} \tag{32}$$

This is again a spatial diffeomorphism along the electric field-dependent vector field

这再次是沿依赖电场的矢量场的空间微分同胚

$$v^a(f,g) = \left[\frac{E_j^a E_l^b \delta^{jl}}{Q^{4-2w}}\omega_b(f,g)\right] \tag{33}$$

As outlined in [97], it can be quantized on $\mathcal{D}$ at $w = 1$; see sections ”Regulator Removal from the Hamiltonian Constraint and Operator Topologies” and ”Commutator Algebra, Closure, and Anomalies”, in particular (24). Basically, the first term becomes a local finite diffeomorphism along a vector field non-vanishing in the vicinity of a vertex that coincides with the tangent vector field on an edge, while ω becomes a discrete derivative along another edge of the graph of a SNWF. This idea of using local, edge tangent field-dependent diffeomorphisms first is described in [96] is also a central ingredient of the work [73, 122, 123] which there is applied also to C_E^G itself.

正如文献 [97] 所述，它可在 $w = 1$ 处于 $\mathcal{D}$ 上量子化；参见章节“哈密顿约束的正则去除与算子拓扑”以及“对易代数、闭包与反常”，特别是式 (24)。基本上，第一项对应顶点附近非零向量场的局部有限微分同胚，该向量场与一条边上的切向量场重合，而 ω 变为自旋网函数图上沿另一条边的离散导数。这种优先使用局部、边切向量场依赖的微分同胚的思想最早记载于文献 [96]，也是工作 [73, 122, 123] 的核心组成部分，在该工作中它还被应用到 C_E^G 本身。

The choice of density weight taken in [73, 122, 123] is $w = \frac{4}{3}$. To see why, note that then

在 [73, 122, 123] 中选取的密度权重为 $w = \frac{4}{3}$ 。欲知缘由，请注意此时

$$u_j^a(f) = \frac{f\left[\varepsilon^2 E_j^a\right]}{\left[\varepsilon^3 Q\right]^{2/3}}, v^a(f, g) = \frac{\left[\varepsilon^2 E_j^a\right]\left[\varepsilon^2 E_l^b\right]\delta^{jl}}{\left[\varepsilon^3 Q\right]^{4/3}}; \tag{34}$$

thus, as envisaged, they can be independently quantized using the technology of section ”Inverse Powers of E and Quantization Ambiguities”, contribute only at the vertices of a graph, and will yield single and double sums, respectively, over its adjacent edges because $\varepsilon^2 E$ becomes a flux operator. Then in a Riemann sum approximation of $C_E^G(f), K_E^G(f, g)$, we encounter the combination $\frac{1}{\varepsilon}\left[F\varepsilon^2\right]\cdot\left[E\varepsilon^2\right]$. A similar combination appears in the spatial diffeomorphism constraint and can be approximated by something close to a finite diffeomorphism times the singular factor $1/\varepsilon$ which prevents this object to have a limit as an operator on $\mathcal{D}$ with respect to the topology of section ”Regulator Removal from the Hamiltonian Constraint and Operator Topologies”.

因此正如预期，它们可以利用“E 的逆幂与量子化歧义”章节的技术独立量子化，仅在图的顶点处产生贡献，并且由于 $\varepsilon^2 E$ 成为通量算符，最终分别得到对相邻边的单重求和与双重求和。随后在 $C_E^G(f), K_E^G(f, g)$ 的黎曼和近似中，我们得到组合项 $\frac{1}{\varepsilon}\left[F\varepsilon^2\right]\cdot\left[E\varepsilon^2\right]$。类似组合出现在空间微分同胚约束中，可以近似为有限微分同胚乘上奇异因子 $1/\varepsilon$ ，这使得该对象无法成为“哈密顿约束的正则去除与算子拓扑”章节所述拓扑下 $\mathcal{D}$ 上的有极限算子。

We consider now the habitat construction. The precise definition of that space is quite complicated; it distinguishes between ”non-degenerate” vertices (at least tri-valent, each triple of edges has non-co-planar tangents at the vertex) and more singular vertices (”kinks”) for whose treatment a Euclidian background metric h and associated Riemann normal coordinates are employed and takes care of graph symmetries [11], the semi-analytic structure [42,76], and the propagation heredity of section ”Solutions and Propagation”. We therefore just sketch the main ideas; in particular, we neglect the fact that the action of the constraint consists of a propagation part similar to [92, 94] which does not play any role in the commutator calculation [44] and an electric diffeomorphism piece on which we focus solely, oversimplify the formulas, and refer the reader to the complete and rigorous case-by-case analysis [73, 122, 123]. We mention that [73, 122, 123] use the

RS volume operator (which has a smaller kernel than the AL volume and in particular does not vanish on tri-valent kinks which are produced by the electric shift-generated diffeomorphisms) and Tikhonov regularization which allow a more convenient expansion of SNWF into volume eigenfunctions as compared to using the AL volume operator and Poisson bracket identities to treat inverse volume powers as it is employed in [92,94]; see section ”Inverse Powers of E and Quantization Ambiguities”.

现在我们讨论居住空间构造。该空间的精确定义十分复杂；它区分了“非退化”顶点 (至少三价，每个边三元组在顶点处的切向量不共面) 和更奇异的顶点 (“扭折”)，处理后者需要用到欧几里得背景度量 h 和相关黎曼法坐标，同时还要考虑图对称性 [11]、半解析结构 [42, 76] 以及“解与传播”章节的传播遗传性。因此我们仅勾勒核心思想: 具体来说，我们忽略约束作用由类似 [92, 94] 的传播部分 (它在对易计算中不起作用 [44]) 和我们唯一关注的电微分同胚部分组成这一事实，简化了公式，并建议读者参阅完整严谨的逐例分析 [73, 122, 123]。我们注意到 [73, 122, 123] 采用了 RS 体积算符 (其核比 AL 体积算符更小，特别地，它在电移位生成微分同胚产生的三价扭折上不为零) 和吉洪诺夫正则化，与文献 [92, 94] 中使用 AL 体积算符和泊松括号恒等式处理逆体积幂的方法相比，更便于将自旋网函数展开为体积本征函数；参见“E 的逆幂与量子化歧义”章节。

Roughly speaking and not considering graphs with kinks, a basis element of the habitat is a sum over SNWF with fixed spins and intertwiners over all graphs of a given diffeomorphism class B without graph symmetries and kinks which has coefficients that depend on a function $F : \sigma^{|V(\gamma)|} \to \mathbb{R}$. Formally,

粗略来说，不考虑带扭折的图时，居住空间的一组基元素是对固定自旋和交织元的自旋网函数求和，遍历给定微分同胚类 B 中所有无图对称性和扭折的图，求和系数依赖于函数 $F : \sigma^{|V(\gamma)|} \to \mathbb{R}$ 。形式上，

$$l_{B,F;j,\iota} = \sum_{\gamma \in B} \left[F\left(\{v\}_{v \in V(\gamma)} \right) \right] < T_{\gamma,j,\iota}, > \tag{35}$$

which is very much like the proposal of [44] just that instead of the general function F on the respective $V(\gamma)$ in [73, 122, 123] , one considers the special function F given by the product over vertices of the same function f . The subspace $\mathcal{D}_1^*$ of $\mathcal{D}^*$ defined by those special functions F is not preserved by the constraint action; however, its range $\mathcal{D}_2^*$ lies also in the domain of the constraint action so that commutators can be computed. We skip those details and consider now a one-parameter group φ_t^u of spatial diffeomorphisms generated by the vector field u and the quantity $t^{-1}\left[\widetilde{U}(\varphi_t^u) - 1\right]$. As $\widetilde{U}$ is not weakly continuous on $\mathcal{H}$, the limit $t \to 0$ cannot be computed there. On the other hand, the dual action gives (we drop the spin and intertwiner labels)

这与文献 [44] 的提议非常相似，区别仅在于: 此处并未对 [73, 122, 123] 中各自的 $V(\gamma)$ 采用一般函数 F ，而是考虑由同一函数 f 在顶点处求乘积得到的特殊函数 F 。由这些特殊函数 F 定义的 $\mathcal{D}^*$ 子空间 $\mathcal{D}_1^*$ 在约束作用下不封闭；但它的值域 $\mathcal{D}_2^*$ 仍属于约束作用的定义域，因此可以计算对易子。我们略去这些细节，接下来考虑由向量场 u 和量 $t^{-1}\left[\widetilde{U}(\varphi_t^u) - 1\right]$ 生成的单参数空间微分同胚群 φ_t^u 。由于 $\widetilde{U}$ 在 $\mathcal{H}$ 上不是弱连续的，无法在该处计算极限 $t \to 0$ 。另一方面，对偶作用给出 (我们省略自旋和 intertwinner 标号)

$$\widetilde{U}'(\varphi_t^u)\, l_{B,F} = l_{B,(\varphi_t^u)^* F} \tag{36}$$

where the $\varphi(B) = B$ was used. Thus, we can take the derivative with respect to t provided that F is at least C^1 and obtain

其中用到了 $\varphi(B) = B$ 。因此，只要 F 至少为 C^1 ，我们就可以对 t 求导，得到

$$iD'[u]\, l_{B,F} := \left[\frac{d}{dt}\widetilde{U}'(\varphi_t^u)\, l_{B,F}\right]_{t=0} = l_{B,u[F]}, u[F]\left(\{v\}_{v\in V(\gamma)}\right)$$

$$:= \sum_{v\in V(\gamma)} u^a(v)\,\frac{\partial F}{\partial v^a} \tag{37}$$

This yields a representation of spatial diffeomorphisms on this habitat.

这就得到了空间微分同胚在该栖息空间上的一个表示。

Turning to the regulated Euclidian vacuum Hamiltonian constraint $C_E^G[f]$, using the analogy with spatial diffeomorphisms, one quantizes it such that in the action on a SNWF over some graph γ_0 , one can essentially replace the electric shift (31) by fu^0 where u^0 is a vector field defined by γ_0 (similar to (24) with local support in the vicinity of vertices (more precisely, it is a sum of such terms)) and the remainder in analogy to a spatial diffeomorphism. Then one obtains, leaving out many details,

转而讨论正规化欧几里得真空哈密顿约束 $C_E^G[f]$ ，利用空间微分同胚的类比，我们对其进行量子化: 当作用在某图 γ_0 上的自旋网波函数时，本质上可以将电移 (31) 替换为 fu^0 ，其中 u^0 是由 γ_0 定义的向量场 (与 (24) 类似，局域支撑在顶点邻域，更准确地说，它是这类项的和)，剩余部分类比空间微分同胚。省略大量细节后，我们得到

$$\left[\left(C_E^G[f]\right)' l_{B,F}\right]\left[T_{\gamma_0}\right] := \lim_{\varepsilon\to 0}\left[\left(C_E^G[f]\right)'_{\varepsilon} l_{B,F}\right]\left[T_{\gamma_0}\right] = \left[D'[fu_0]\, l_{B,F}\right]\left[T_{\gamma_0}\right]$$

$$= l_{B,(fu_0)[F]}\left[T_\gamma\right] \tag{38}$$

It follows for the commutator

由此可得对易子

$$\left[\left[\left(C_E^G[f]\right)', \left(C_E^G[g]\right)'\right] l_{B,F}\right]\left[T_{\gamma_0}\right] = l_{B,[[fu^0,gu^0]u^0F]}\left[T_\gamma\right]$$

$$= l_{B,((f(u^0[g])-g(u^0[f])u^0)[F]}\left[T_\gamma\right] \tag{39}$$

On the other hand, for $K_E^G(f,g)$, one quantizes it such that in the action on a SNWF over γ_0 , the electric field-dependent vector (33) is essentially given by $(\,fu^0[g] -\, g\,u^0[f])\,u^0$ (more precisely, it is a sum of such terms). Then indeed we obtain closure, again leaving out many details,

另一方面，对于 $K_E^G(f,g)$ ，我们将其量子化: 当作用在 γ_0 上的自旋网波函数时，依赖电场的向量 (33) 本质上由 $(\,fu^0[g] -\, g\,u^0[f])\,u^0$ 给出 (更准确地说，它是这类项的和)。同样省略大量细节，我们确实得到了闭代数,

$$\left[\left(K_E^G(f,g)\right)' l_{B,F}\right][T_{\gamma_0}] := \lim_{\varepsilon\to 0}\left[\left(K_E^G(f,g)\right)'_{\varepsilon} l_{B,F}\right][T_{\gamma_0}]$$

$$= \left[\left[\left(C_E^G[f]\right)', \left(C_E^G[g]\right)'\right] l_{B,F}\right][T_{\gamma_0}] \tag{40}$$

We close this subsection with a couple of remarks:

我们以几条注记结束本小节:

1. As mentioned, the close algebraic relation between D_E^G, C_E^G was maximally exploited to arrive at this result. Comparing, e.g., to the case of a (uncharged) scalar field $D_a^S = \pi\phi_{,a}, 2C^S = Q^{w-1}\left[\frac{\pi^2}{Q} + Q\left[q^{ab}\phi_{,a}\phi_{,b} + V(\phi)\right]\right.$, it is no longer true that both D^S and C^S are linear in the "curvature" $\phi_{,a}$ as it is true for D_E^G, C_E^G and which is essential for the above to work. Similar remarks hold for the YM and fermion (with mass terms) contributions. Furthermore, the density weight $w = 4/3$ is also geared just to the C_E^G contribution and does not work for the others.

1. 如前所述，我们最大限度利用了 D_E^G, C_E^G 之间紧密的代数关系才得到该结果。例如，与 (不带电) 标量场 $D_a^S = \pi\phi_{,a}, 2C^S = Q^{w-1}\left[\frac{\pi^2}{Q} + Q\left[q^{ab}\phi_{,a}\phi_{,b} + V(\phi)\right]\right.$ 的情况相比，此处不再像 D_E^G, C_E^G 那样满足 D^S 和 C^S 都对“曲率” $\phi_{,a}$ 线性，而这正是上述构造成立的核心。类似的结论也适用于杨-米尔斯和费米子 (带质量项) 贡献。此外，密度权 $w = 4/3$ 也仅适配 C_E^G 贡献，对其他贡献不成立。

2. There is a reduction of certain ambiguities because the quantization of C_E^G follows closely that of D_E^G . Thus, $\left[\widetilde{U}(\varphi_{v,e}) - 1\right] T_{\gamma_0}$ when expanded in terms of holonomies g around the loops $\alpha_{v,e,e'} := e \circ \varphi_{v,e}(e)^{-1} \circ \varphi_{v,e}(e') \circ e'^{-1}$ based at v at which e, e' are adjacent and where $\varphi_{v,e}$ is a diffeomorphism along a vector field with support close to v and which on e coincides with the tangent of e (in particular, $\varphi_{v,e}(e) \subset e$; see (24)) gives an expression of the form (π some spin representation)

2. 由于 C_E^G 的量子化与 D_E^G 的量子化高度一致，某些歧义得到了约化。因此，当 $\left[\widetilde{U}(\varphi_{v,e}) - 1\right] T_{\gamma_0}$ 按照基于 v 的环 $\alpha_{v,e,e'} := e \circ \varphi_{v,e}(e)^{-1} \circ \varphi_{v,e}(e') \circ e'^{-1}$ 周围的全纯和 g 展开时——其中 e, e' 相邻，$\varphi_{v,e}$ 是支撑靠近 v 的向量场上的微分同胚，且在 e 上与 e 的切线重合 (特别是 $\varphi_{v,e}(e) \subset e$；见 (24))——会得到形如 ($\pi$ 某个自旋表示) 的表达式

$$\pi(gh_{e'}) - \pi(h_{e'}) = \left[\pi\left(1_2 + \mathrm{Tr}(g\tau_j)\tau_j + \ldots\right) - 1_\pi\right]\pi(h_{e'})$$

$$\mathrm{Tr}(g\tau_j) X_\pi^j \pi(h_{e'}) + \ldots = \mathrm{Tr}(g\tau_j)\frac{d}{dt}\pi\left(e^{t\tau_j}h_{e'}\right) + \ldots \tag{41}$$

thus uses the loop holonomy in the spin 1/2 representation and a right invariant vector field (quantization of the flux operator). Thus, it confirms the choice of spin 1/2 made in [92, 94] and removes some of the ambiguities pointed out in [80]. On the other hand, inverse volume ambiguities and loop size type of ambiguities are still present. One also has to invoke an at least minimal amount of an additional type of freedom, which in earlier works was suppressed by the principle of naturalness consisting, e.g., in the treatment of kink contributions and addition of higher-order terms (in ε) which ensure both propagation and closure as discussed in [73, 122, 123] . Finally, the choice of a habitat itself is equivalent to a choice of representation, and while the habitats do not come equipped with a (Hilbert space) topology, we expect that different choices of habitats lead to unitarily inequivalent representations of the algebra of observables in the resulting physical Hilbert space obtained by supplying a Hilbert space structure to the space of solutions. Thus, further reduction of ambiguities, e.g., by renormalization methods, is still necessary.

> 因此使用了自旋 1/2 表示中的圈全纯和右不变向量场 (通量算符的量子化)。这证实了 [92, 94] 中对自旋 1/2 的选择，消除了文献 [80] 指出的部分歧义。另一方面，逆体积歧义和圈尺寸类歧义仍然存在。还必须引入至少最小程度的额外自由度，这类自由度在早期工作中被自然性原理抑制；例如自然性原理体现在对扭折贡献的处理，以及添加 (ε 中的) 高阶项，正如 [73, 122, 123] 中讨论的，这些项同时保证了传播性和封闭性。最后，栖息地本身的选择等价于表示的选择，尽管栖息地本身没有配备 (希尔伯特空间) 拓扑，但我们预期，不同的栖息地选择会导致，在为解空间添加希尔伯特空间结构后得到的物理希尔伯特空间中，可观测量代数出现幺正不等价的表示。因此，仍然需要进一步减少歧义，例如通过重整化方法。

3. It is quite remarkable that for the first time, one can make the details work out to close the algebra without anomalies in a technically precise sense (since the construction is diffeomorphism covariant, also the dual commutators between spatial diffeomorphisms and the Hamiltonian constraints work out as expected), in particular that in the actual sums involved, the double sum of the commutator (39) reduces to the single sums (40) with exactly the right coefficients. Note that in our simplified exposition, the action (38) suggests that all solutions to the spatial diffeomorphism constraint with no kinks and graph symmetries (corresponding to constant F) are already in the kernel of $\left(C_E^G[f]\right)'$. However, the proper treatment establishes that those states are not even in the domain of the dual Hamiltonian constraint. Thus, in agreement with one's intuition, the two constraints have individual solutions not in the joint kernel.

> 3. 非常值得注意的是，我们首次能够让细节推导自洽，在技术精确意义上无反常地封闭代数 (由于构造是微分同胚协变的，空间微分同胚与哈密顿约束之间的对偶对易子也符合预期)，尤其值得一提的是，在实际涉及的求和中，对易子 (39) 的二重求和约化为单求和 (40)，且系数完全正确。注意在我们的简化阐释中，作用量 (38) 表明，所有不存在扭折和图对称性的空间微分同胚约束解 (对应常数 F) 已经位于 $\left(C_E^G[f]\right)'$ 的核中。但恰当的处理证明，这些态甚至不在对偶哈密顿约束的定义域内。因此，符合我们的直觉，两个约束各自存在不在公共核中的解。

4. An interesting observation is the following [10]: Since the action of C_E^G is so close to a spatial diffeomorphism, one may wonder whether it can be supplied with a corresponding geometrical interpretation. This is indeed the case: Note first that D_E^G generates ordinary Lie derivatives $\mathcal{L}_u$ along ordinary vector fields u . These derivatives are not Gauß gauge covariant when acting on Lie algebra-valued fields. However, the combination $\widehat{D}_E^G = D_E^G + A^j G_j$ used in (30) does and generates gauge-covariant Lie derivatives $\mathcal{L}_u$. Likewise, the Euclidian vacuum Hamiltonian constraint generates generalized gauge-covariant Lie derivatives $\widehat{\mathcal{L}}_{\vec{u}}$ where now $\vec{u}_j^a$ is a Lie algebra-valued vector field (namely, the electric shift). The definition of these Lie derivatives differs from the ordinary one by replacing ∂ in the formula for $\mathcal{L}$ by $\mathcal{D} = \partial + A$ when ∂ would act on a Lie algebra-valued field in order to maintain gauge covariance. These derivatives can be extended to vector fields of any internal tensor degree forming a generalized (open) algebra. Note however that u_j^a is phase space dependent and not a test field. This geometric interpretation supplies further motivation to quantize C_E^G in close analogy to D_E^G . See also the next subsection for a concrete quantization of this algebra in the $\mathrm{U}(1)^3$ model.

4. 一个有趣的结论如下 [10]: 由于 C_E^G 的作用非常接近空间微分同胚，我们自然会思考它是否存在对应的几何诠释。事实确实如此: 首先注意，D_E^G 生成沿普通矢量场 u 的普通李导数 $\mathcal{L}_u$。当这类李导数作用在取值为李代数的场上时，不满足高斯规范协变性。但 (30) 中使用的组合 $\widehat{D}_E^G = D_E^G + A^j G_j$ 满足这一性质，并且可以生成规范协变李导数 $\mathcal{L}_u$。同理，欧几里得真空哈密顿约束生成广义规范协变李导数 $\widehat{\mathcal{L}_{\vec{u}}}$，此时 $\vec{u}_j^a$ 是一个取值为李代数的矢量场 (即电移)。为了维持规范协变性，当 ∂ 作用在取值为李代数的场上时，这类李导数的定义与普通定义的区别在于，将 $\mathcal{L}$ 公式中的 ∂ 替换为 $\mathcal{D} = \partial + A$。这类李导数可以推广到任意内张量次数的矢量场，形成一个广义 (开) 代数。但需要注意，u_j^a 依赖于相空间，并非测试场。这种几何诠释为按照与 D_E^G 高度相似的方式量子化 C_E^G 提供了进一步动机。关于该代数在 $\mathrm{U}(1)^3$ 模型中的具体量子化，参见下一小节。

Quantum Non-degeneracy

量子非简并性

The theory laid out so far has revealed that to arrive at a representation of $\mathfrak{h}$, one has to take non-standard steps, either by looking at non-standard operator topologies when an implementation on $\mathcal{D}$ is intended as sketched in section ”Regulator Removal from the Hamiltonian Constraint and Operator Topologies” or by using non-standard density weights when an implementation on subspaces (habitats) of $\mathcal{D}^*$ is intended as sketched in the previous subsection. One may wonder how these complications arise and how non-standard constructions can be avoided.

迄今构建的理论表明，要得到 $\mathfrak{h}$ 的表示，必须采用非标准步骤: 若如“哈密顿约束的正则子去除与算子拓扑”一节所述，计划在 $\mathcal{D}$ 上实现，则需采用非标准算子拓扑；若如前一小节所述，计划在 $\mathcal{D}^*$ 的子空间 (生境) 上实现，则需采用非标准密度权重。人们不禁会问，这些复杂问题因何产生，又该如何避免非标准构造。

We note that in both implementations, the density weight w is lower than two. Therefore, in the classical theory, we have $\{C[f], C[g]\} = -D_a\left[Q^{w-2}E_j^a E_k^b \delta^{jk} \quad (fg_{,b} - gf_{,b})\right]$ depending on a negative power of $Q = |\det(E)|^{1/2}$. It is an implicit assumption of classical GR that Q is nowhere vanishing, i.e., non-degenerate. We may consider a relaxation of this assumption in quantum GR, but we then expect problems in constructing a representation of $\mathfrak{h}$ due to the negative power of Q. In fact, the dense domain $\mathcal{D}$ of $\mathcal{H}$ defined by the span of SNWF is such that the volume operator $V(R) = \int_R d^3xQ$ vanishes on any element of $\mathcal{D}$ unless R intersects a Lebesgue measure zero set. In that sense, the quantum geometries described by SNWF are quantum degenerate almost everywhere. It is for this reason that one had to quantize inverse powers of Q carefully as it was sketched in section ”Inverse Powers of E and Quantization Ambiguities” with the result that inverse powers of $V(R)$ annihilate a SNWF unless R contains at least one non-zero volume vertex of the underlying graph. Still this brings us into the awkward situation that we try to implement the constraints on a quantum domain whose semiclassical limit is classically forbidden and on which the classical $\mathfrak{h}$ is ill-defined.

我们注意到，在两种实现中，密度权重 w 都小于 2。因此在经典理论中，$\{C[f], C[g]\} = -D_a\left[Q^{w-2}E_j^a E_k^b \delta^{jk} \quad (fg_{,b} - gf_{,b})\right]$ 依赖于 $Q = |\det(E)|^{1/2}$ 的负幂次。经典广义相对论有一个隐含假设: Q 处处非零，也就是非简并的。我们可以在量子广义相对论中放宽这一假设，但由于 Q 存在负幂次，我们在构造 $\mathfrak{h}$ 的表示时预期会遇到问题。事实上，由自旋结网波张成的 $\mathcal{H}$ 的稠定域 $\mathcal{D}$ 满足: 体积算符 $V(R) = \int_R d^3xQ$ 在 $\mathcal{D}$ 的任意元素上都为零，除非 R 与一个勒贝格零测集相交。从这个意义上说，自旋结网波描述的量子几何在几乎处处都是量子简并的。正因如此，人们必须仔细对 Q 的逆幂次进行量子化，这一点在“E 的逆幂次与量子化歧义”一节中已有概述，得到的结论是: 除非 R 在其底图中包含至少一个非零体积顶点，否则 $V(R)$ 的逆幂次会湮灭自旋结网波。即便如此，我们仍陷入这样一种尴尬的处境: 我们试图在一个量子域上实现约束，而该量子域的半经典极限是经典理论不允许的，且经典 $\mathfrak{h}$ 在这个域上是不良定义的。

This explains why one is forced into the above non-standard steps as follows: The classical constraints $D[u], C[f]$ are integrals over σ of the densities $D_a(x)$ and $C(x)$, respectively, and in order to define the operator corresponding to $C[f]$, we used a standard point-splitting regularization based on a Riemann sum approximation $C_\varepsilon[f] = \sum_\Delta f(p_\Delta) C_\Delta$ with $C_\Delta = \left[\int_\Delta \quad d^3x \quad C(x) \right]$. Here, $p_\Delta \in \Delta$, the cells Δ have coordinate volume ε^3 , and there are an order of $N_\varepsilon := \varepsilon^{-3}$ terms if σ is compact (in the non-compact case, consider f of compact support) which makes sure that the sum converges to a non-zero limit (the integral) as $\varepsilon \to 0$. Now in the quantum theory, we quantize C_Δ on a SNWF T_γ , and eventually, at most $N_\gamma := |V(\gamma)|$, cells contribute as $\varepsilon \to 0$. Due to diffeomorphism invariance, the norm $\left\|C_\varepsilon[f] T_\gamma\right\|$ is finite, non-vanishing, and ε independent if we define $p_\Delta = v$ when $\Delta \cap V(\gamma) = \{v\}$. However, for the analogous quantization $K(f,g)$ of the Poisson bracket $\{C(f), C(g)\}$ sketched in section ”Commutator Algebra, Closure, and Anomalies”, the limit of the norm vanishes for $w = 1$ unless f, g have discontinuities. This is because $K_\varepsilon(f,g)$ again has an order of N_γ contributions but now depends on the finite differences schematically denoted as $f(v)\, g(v + \varepsilon \dot{e}(v)) - g(v)\, f(v + \varepsilon \dot{e}(v))$ where e is an edge of γ adjacent to v . This is also the underlying reason why the algebra on the habitat [44] is Abelian and why in the previous section a different density weight was considered so that instead the combination $[f(v)\, g(v + \varepsilon \dot{e}(v)) - g(v)\, f(v + \varepsilon \dot{e}(v))]/\varepsilon$ results.

这就解释了为何我们不得不采用上述非标准步骤，具体如下: 经典约束 $D[u], C[f]$ 分别是密度 $D_a(x)$ 和 $C(x)$ 在 σ 上的积分，为了定义对应于 $C[f]$ 的算符，我们采用了基于黎曼和近似 $C_\varepsilon[f] = \sum_\Delta f(p_\Delta) C_\Delta$ 的标准点分裂正则化，其中涉及 $C_\Delta = \left[\int_\Delta \quad d^3x \quad C(x) \right]$。此处，在 $p_\Delta \in \Delta$ 中，单元 Δ 的坐标体积为 ε^3 ，若 σ 是紧致的 (非紧致情况则考虑 f 具有紧致支集)，共有约 $N_\varepsilon := \varepsilon^{-3}$ 项，这确保了当 $\varepsilon \to 0$ 时，和收敛到非零极限 (即该积分)。现在在量子理论中，我们在 SNWF 基 T_γ 上对 C_Δ 量子化，最终当 $\varepsilon \to 0$ 时，至多有 $N_\gamma := |V(\gamma)|$ 个单元产生贡献。由于微分同胚不变性，当我们在 $\Delta \cap V(\gamma) = \{v\}$ 条件下定义 $p_\Delta = v$ 时，范数 $\left\|C_\varepsilon[f] T_\gamma\right\|$ 是有限、非零且与 ε 无关的。然而，对于“对易子代数、闭合性与反常”一节中简述的泊松括号 $\{C(f), C(g)\}$ 的类似量子化方案 $K(f,g)$ ，除非 f, g 存在不连续性，否则当 $w = 1$ 时范数的极限为零。这是因为 $K_\varepsilon(f,g)$ 同样有约 N_γ 项贡献，但现在它依赖于由 $f(v)\, g(v + \varepsilon \dot{e}(v)) - g(v)\, f(v + \varepsilon \dot{e}(v))$ 概要表示的有限差分，其中 e 是 γ 中与 v 邻接的一条边。这也是为什么栖居域上的代数 [44] 是阿贝尔代数，以及为什么上一节我们考虑了不同的密度权，最终得到组合 $[f(v)\, g(v + \varepsilon \dot{e}(v)) - g(v)\, f(v + \varepsilon \dot{e}(v))]/\varepsilon$ 的根本原因。

Thus, while in the classical theory $K(f,g)$ is always non-vanishing for smooth f, g and any density weight in the quantum theory, we obtain this peculiar and awkward behavior. The source of the trouble is the quantum degeneracy of the domain $\mathcal{D}$ or the entire representation. If one would work in a representation which is quantum non-degenerate, then the number of contributing C_Δ would be an increasing function of ε^{-1} , and there would be a chance that this awkward behavior can be avoided.

因此，尽管在经典理论中，对于光滑的 f, g，$K(f, g)$ 在任意密度权重下始终非零，但在量子理论中我们却得到了这种奇特且别扭的行为。问题的根源在于定义域 $\mathcal{D}$ 或整个表示存在量子简并。如果在量子非简并的表示中工作，那么贡献 C_Δ 的数量会是 ε^{-1} 的增函数，就有可能避免这种别扭的行为。

In [17, 18, 101] , a quantum non-degenerate representation for $\mathrm{U}(1)^3$ quantum gravity (which is a toy model very close to Euclidian vacuum GR and presents a consistent deformation in terms of a small Newton constant of Euclidian vacuum GR in the sense of [19]) was found which allows for an anomaly-free representation of the Bergmann-Komar "group" $\mathfrak{H} := \exp(\mathfrak{h})$ (the exponentiation avoids different treatments of finite diffeomorphisms and infinitesimal Hamiltonian constraints; the BK group is here defined as the Lie group defined by the true Lie algebra obtained by taking commutators of all hypersurface deformations (universal enveloping algebra of [67])) for any density weight, on the corresponding $\mathcal{D}$ without using habitats. The model is quantum integrable, i.e., a physical Hilbert space representation can be found, because in contrast to the $\mathrm{SU}(2)$ theory, the $\mathrm{U}(1)^3$ has the property that the Hamiltonian constraint preserves the momentum polarization of the phase space in the sense of geometric quantization [128]. This can be considered as a quantization of the generalized gauge-covariant derivatives of [10] (see previous section) to all orders. The model can be extended by a cosmological constant term. An intriguing idea is that one could perhaps define the full SU(2) theory in terms of perturbation theory around this integrable theory as spelled out in [19]. Note that while the standard LQG representation and the non-degenerate representation for $\mathrm{U}(1)^3$ are different, they are still very similar in the sense that they use Narnhofer-Thirring-type representations and thus much of the technology developed for LQG can be transferred. In particular, all the results for $\mathrm{U}(1)^3$ were obtained by exactly the same steps as in the present LQG representations with only minor modifications due to the different gauge groups in place.

在 [17, 18, 101] 中，研究者找到了 $\mathrm{U}(1)^3$ 量子引力的一种量子非简并表示 ($\mathrm{U}(1)^3$ 量子引力是非常接近欧几里得真空广义相对论的玩具模型，且按照文献 [19] 的意义，它给出了欧几里得真空广义相对论以小牛顿常数展开的一致形变)，该表示可以在对应 $\mathcal{D}$ 上不对任何密度权重使用生境 (habitats), 就能得到无反常的 Bergmann-Komar“群” $\mathfrak{H} := \exp(\mathfrak{h})$ 表示 (指数化避免了对有限微分同胚和无穷小哈密顿约束的区别处理；此处 BK 群定义为由所有超曲面形变取对易得到的真实李代数构造的李群，即文献 [67] 的泛包络代数)。该模型是量子可积的，也就是说可以找到物理希尔伯特空间表示，因为和 $\mathrm{SU}(2)$ 理论不同，$\mathrm{U}(1)^3$ 具备哈密顿约束在几何量子化 [128] 的意义下保持相空间动量极化的性质。这可以看作是文献 [10] 中广义规范协变导数 (见上一节) 在所有阶下的量子化。该模型可以扩展加入宇宙学常数项。一个引人关注的思路是，或许可以按照文献 [19] 给出的方案，围绕这个可积理论用微扰论定义完整的 SU(2) 理论。值得注意的是，尽管标准 LQG 表示和 $\mathrm{U}(1)^3$ 的非简并表示并不相同，但二者本质上非常相似，都采用 Narnhofer-Thirring 型表示，因此为 LQG 开发的大部分技术都可以直接迁移使用。具体来说，$\mathrm{U}(1)^3$ 的所有结果都是通过和现有 LQG 表示完全相同的步骤得到的，仅仅因为规范群的不同做了微小修改。

This completely solvable model [101], which is quantized following step by step the full arsenal of technologies developed for LQG, may serve as a proof of principle that making quantum degeneracy a prerequisite for representing $\mathfrak{h}$ or $\mathfrak{H}$ may be a promising direction to make progress as spelled out in detail in [100]. To arrive at such representations systematically or constructively, renormalization methods suggest themselves because as sketched in section "Renormalization" in the renormalization program one works, at each finite resolution, by definition with a dense set of non-degenerate states for that resolution and thus at infinite resolution, one expects quantum non-degeneracy to be inherited.

这个完全可解模型 [101] 是一步步沿用为 LQG 开发的全套技术完成量子化的，它可以作为原理验证，说明正如文献 [100] 详细阐述的那样，将量子非简并作为表示 $\mathfrak{h}$ 或 $\mathfrak{H}$ 的前提，是推动研究进展的可行方向。为了系统或构造性地得到这类表示，重整化方法是自然的选择，因为正如“重整化”一节所述，在重整化方案中，每个有限分辨率下，按照定义处理的都是该分辨率下的一组非简并态的稠密集合，因此可以预期在无穷分辨率下依然会保留量子非简并性。

Constraining Before Quantization

量子化前约束化

In the reduced approach, GR has been cast into the framework of an ordinary Hamiltonian theory. Therefore, the quantization requires to find representations of the observable algebra, and the quantum dynamics can be implemented by quantizing the physical Hamiltonian. It is also possible to perform only a partial reduction typically with respect to the Hamiltonian constraint only and then solve the remaining constraint via Dirac quantization. For instance, in the existing models [39,50,55,57,68], the Gauß constraint is solved using Dirac quantization, and either one or four, respectively, dust or scalar fields are used as reference fields for the Hamiltonian and spatial diffeomorphism constraint, respectively. A classification of the different types of models can be found in [56], where a generic Lagrangian was analyzed that for appropriate choices of the involved parameters encodes the existing dust models in the literature [22, 27, 72] based on seminal work of Kuchař et al. These kinds of models, denoted as type I models in [56], all have in common that one couples six to eight additional fields to gravity yielding to a second-class system. Performing a symplectic reduction with respect to the second-class constraints leads to a system that involves four (three) additional (null) dust fields that can, in the case of dust, be used as a dynamical reference frame and for null dust as a dynamical spatial reference frame. The remaining Hamiltonian, spatial diffeomorphism, and Gauß constraints are first class so that one can apply the usual construction of observables in the framework of the relational formalism in these models. A further type I model not involved in [56] can be found in [57] where one Klein-Gordon scalar field and six additional scalar fields were minimally coupled to GR. The second class of models, denoted as type II in [56], involves a matter Lagrangian based on one scalar field, and hence, within such models, only a partial reduction of the classical constraints can be achieved. Examples for such models can be found in [39,111]. As mentioned above, the choice of reference matter is strongly guided by the aim to obtain a manageable observable algebra in the reduced models. As we will discuss below for all these models, the reduced quantization program can be completed, and the final physical Hilbert space is known. Because the reduced models always involve a true Hamiltonian that is non-vanishing, they are also of advantage if one wants to generalize to open quantum systems where one often starts with a given Hamiltonian of the total system. The first steps in this direction using the relational formalism and a reduced quantization can be found, for instance, in [41].

在约化方案中，广义相对论已被纳入普通哈密顿理论框架。因此，量子化需要找到可观测量代数的表示，且量子动力学可通过对物理哈密顿量量子化来实现。也可以仅做部分约化，通常是仅对哈密顿约束做部分约化，之后通过狄拉克量子化求解剩余约束。例如，在已有模型 [39,50,55,57,68] 中，高斯约束通过狄拉克量子化求解，分别使用 1 个或 4 个尘埃场或标量场作为参考场，分别对应哈密顿约束和空间微分同胚约束。不同类型模型的分类可见文献 [56]，该文献分析了一个通用拉格朗日量，对于合适的参数选择，它可以涵盖文献中基于库查尔等人开创性工作的已有尘埃模型 [22, 27, 72] 。这类模型在 [56] 中被标记为 I 型模型，其共同特点是引入 6 到 8 个额外场与引力耦合，得到一个二类系统。对二类约束做辛约化后得到的系统包含四个 (三个) 额外 (零) 尘埃场，对于普通尘埃而言，这些场可作为动力学参考系，对于零尘埃而言，可作为动力学空间参考系。剩余的哈密顿约束、空间微分同胚约束和高斯约束都是一级约束，因此可以在这些模型的关系形式论框架中应用常规的可观测量构造方法。[56] 未收录的另一个 I 型模型可见文献 [57]，该模型将一个克莱因-戈登标量场和六个额外标量场最小耦合到广义相对论。第二类模型在 [56] 中被标记为 II 型模型，包含基于单个标量场的物质拉格朗日量，因此在这类模型中只能对经典约束做部分约化。这类模型的例子可见文献 [39,111]。如上所述，参考物质的选择很大程度上以在约化模型中得到可处理的可观测量代数为目标。正如我们下文对所有这些模型的讨论，约化量子化方案可以完成，且最终的物理希尔伯特空间是已知的。由于约化模型始终包含一个非零的真实哈密顿量，当我们想要推广到开放量子系统时 (开放量子系统通常从总系统的给定哈密顿量出发)，这类模型也更具优势。例如，沿这个方向利用关系形式论和约化量子化的初步工作可见文献 [41]。

Reduced Quantization of Type I Models

I 类模型的约化量子化

In the used notation, (Q^A, P_A) collectively denote all degrees of freedom not related to the reference fields and (T^I, P_I) those of the reference fields, where for type I models, $I = 0, \cdots, 3$ and A labels the remaining degrees of freedom. All variables Q^A, P_A that we want to construct observables of Poisson commute with the chosen reference fields T^I . The corresponding gauge fixing conditions $F_I = T^I - \tau^I$ being linearly in these reference fields also commute with the remaining variables and consequently agree the Dirac bracket with the Poisson bracket in this case. The observable algebra of O_{Q^A}, O_{P_A} thus reads

在本文所用记号中，(Q^A, P_A) 统一表示所有与参考场无关的自由度，(T^I, P_I) 统一表示参考场的自由度，其中对于 I 类模型，$I = 0, \cdots, 3$ 和 A 标记剩余自由度。所有我们要构造可观测量的变量 Q^A, P_A 都与选定的参考场 T^I 泊松对易。对应的规范固定条件 $F_I = T^I - \tau^I$ 关于这些参考场是线性的，因此也与剩余变量对易，在此情况下狄拉克括号与泊松括号一致。因此 O_{Q^A}, O_{P_A} 的可观测量代数可写为

$$\left\{O_{Q^A}\left(\sigma, \tau^0\right), O_{P_B}\left(\sigma', \tau^0\right)\right\} = \lambda \delta^A_B \delta\left(\sigma, \sigma'\right),$$

where $\tau^I = \left(\tau^0, \sigma^j\right), j = 1, 2, 3$ denote the physical temporal and spatial coordinates, λ is a possible coupling parameter that can also be equal to 1, and all remaining Poisson brackets vanish so that the algebra of these observables satisfies standard CAR and AR. Since for type I models O_{Q^A}, O_{P_A} are observables with respect to the Hamiltonian and spatial diffeomorphism constraints, a representation of the observable algebra yields direct access to the physical Hilbert space if one solves the Gauß constraint in the quantum theory. Next to solving the Gauß constraint, one is only interested in those representations for which the physical Hamiltonians of the individual models can be promoted to well-defined operators. Three examples for physical Hamiltonians H can be found in (42) below

其中 $\tau^I = \left(\tau^0, \sigma^j\right), j = 1, 2, 3$ 表示物理时间坐标与空间坐标，λ 是可等于 1 的可能耦合参数，其余所有泊松括号均为零，因此这些可观测量的代数满足标准 CAR 与 AR。由于对于 I 类模型，O_{Q^A}, O_{P_A} 是关于哈密顿约束和空间微分同胚约束的可观测量，若在量子理论中求解高斯约束，可观测量代数的表示可直接得到物理希尔伯特空间。除求解高斯约束外，我们只关心那些能将单个模型的物理哈密顿量提升为良定义算符的表示。下文 (42) 给出了物理哈密顿量 H 的三个示例

$$H = \int_{\mathcal{S}} d^3\sigma H(\sigma) \tag{42}$$

$$H(\sigma) = \sqrt{\left(O_C\right)^2 - Q^{jk} O_{D_j} O_{D_k}}(\sigma) \quad \text{(Brown-Kuchař)}$$

$$H(\sigma) = O_C(\sigma) \quad \text{(Gaussian)}$$

$$H(\sigma) = \sqrt{-2\det(Q)\, O_C + 2\sqrt{\det(Q) \sum_{j=1}^{3} Q^{jj} O_{D_j} O_{D_j}}}, \text{(4 scalar fields)} \tag{57}$$

where $\mathcal{S}$ symbolizes the manifold of physical spatial coordinates σ^j and $Q_{jk} := O_{q_{jk}}$ denotes the observable of the spatial metric understood as a function of the densitized (co-)triads $E.O_C$ and O_{D_j} , respectively, denote the observable of the contribution to Hamiltonian and spatial diffeomorphism constraint, respectively, of the physical sector encoded in O_{Q^A}, O_{P_A} . As one can see, a generic feature of all these models is that the physical Hamiltonian densities $H(\sigma)$ often involve square roots, and inside the square, usually powers of O_C and O_{D_j} together with contractions with the metric as well possible density weight factors occur, where the latter ensure that $H(\sigma)$ is a scalar density of weight one. Due to the simple structure of the observable algebra, one can use the standard LQG representation that was used in section "Constraining After Quantization" for the kinematical Hilbert space here for the physical Hilbert space if one considers in addition SU(2) gauge-invariant SNWF. Although an operator for O_{C_j} does not exist in this representation for the physical Hamiltonians, one only needs to be able to quantize the combination $Q^{jk} O_{D_j} O_{D_k}$ which, as will be explained below, is possible to quantize along the lines how the operator for O_C is constructed. Note that not exactly the same but a similar expression is also involved in the extended master constraint in section "(Extended) Master Constraint". Written as a function of the original kinematical variables Q^A, P_A, T^I, P_I , the observables O_C, O_{D_j} are complicated because already the elementary observables O_{Q^A}, O_{P_A} are in general an infinite power series in the reference fields with phase space-dependent coefficients. However, one can use the properties of the observable map to show that [36, 83, 110, 124, 125]

其中 $\mathcal{S}$ 表示物理空间坐标构成的流形，σ^j 与 $Q_{jk} := O_{q_{jk}}$ 是空间度规的可观测量，理解为密化 (余) 三元组 $E.O_C$ 的函数，O_{D_j} 分别表示编码在 O_{Q^A}, O_{P_A} 中的物理部分对哈密顿约束和空间微分同胚约束贡献的可观测量。不难发现，所有这类模型的一个普遍特征是物理哈密顿密度 $H(\sigma)$ 通常带有平方根，根号内通常是 O_C 和 O_{D_j} 的幂次，搭配度规缩并以及可能的密度权重因子，后者保证 $H(\sigma)$ 是权重为 1 的标量密度。由于可观测量代数结构简单，若额外考虑 SU(2) 规范不变自旋网波函数，就可以将“量子化后约束”一节中用于运动学希尔伯特空间的标准 LQG 表示直接用于此处的物理希尔伯特空间。尽管在该表示中物理哈密顿量的 O_{C_j} 算符不存在，我们只需要对组合 $Q^{jk}O_{D_j}O_{D_k}$ 进行量子化，正如下文将要解释的，可以按照构造 O_C 算符的方式完成量子化。需要注意，在“(推广) 主约束”一节的推广主约束中也涉及一个并不完全相同但结构类似的表达式。若将其写为原始运动学变量 Q^A, P_A, T^I, P_I 的函数，可观测量 O_C, O_{D_j} 形式十分复杂，因为即使是基本可观测量 O_{Q^A}, O_{P_A}，一般也是参考场中以相空间依赖量为系数的无穷幂级数。不过我们可以利用可观映射的性质证明 [36, 83, 110, 124, 125]

$$O_C \simeq C\left(O_{Q^A}, O_{P_A}\right)\ O_{D_j} \simeq D_j\left(O_{Q^A}, O_{P_A}\right),$$

where $\simeq$ denotes a weak equality. Therefore, one can apply the strategy how C was quantized, discussed in section "Complete Regulated Operator" and also here, and quantize $(O_C)^2$ as $\widehat{O}_C^\dagger \widehat{O}_C$ using similar steps. The corresponding formula to (30) involving the Euclidian part of C in the reduced case is given by

其中 $\simeq$ 表示弱等式。因此，可以沿用“完整正则化算符”小节 (也在本文此处讨论过) 中 C 的量子化策略，通过相似步骤将 $(O_C)^2$ 量子化为 $\widehat{O}_C^\dagger \widehat{O}_C$ 。约化情形下，对应式 (30)、包含 C 欧几里得部分的公式由下式给出

$$\left(O_{C_E^G}\right)^2(\sigma) = \frac{\varepsilon^{JKL}(O_F)^J_{k\ell}(O_E)^\ell_K(O_E)^\ell_L\varepsilon^{J'K'L'}(O_F)^{J'}_{k'\ell'}(O_E)^{k'}_{K'}(O_E)^{\ell'}_{L'}}{\det(O_E)}(\sigma)$$

(σ)

$$= [\mathrm{Tr}(B\tau_0)]^2(\sigma),\ B := B^j_{J'}e^J_j\tau_{J'}\tau_J = \frac{1}{2}\varepsilon^{jk\ell}(O_F)^{J'}_{k\ell}e^J_j\tau_{J'}\tau_J$$

(43)

where $\tau_\mu := (\tau_0 := \mathbb{1}, \tau_I = -i\sigma_I)$, with σ_I being the Pauli matrices, e^J_j denotes the co-triad, and indices referring to physical coordinates are labeled by lowercase letters and those referring to internal SU(2) indices by capital letters. In the explicit action of the corresponding operators, the contribution of the curvature O_F involved in B is thus quantized as a loop of holonomy operators that goes along already existing edges of the underlying graph that the SNWF is defined on. For the reason that the representation of the holonomy operator can couple with representations associated with the edges to the trivial representation and this would correspond to an edge that is annihilated, one needs to add corresponding projection operators to the physical Hamiltonian operator $\widehat{H}$ in order to ensure that it is indeed graph preserving [50]. The contribution of $Q^{jk}O_{D_j}O_{D_k}$ in terms of the A, E variables reads

其中 $\tau_\mu := (\tau_0 := \mathbb{1}, \tau_I = -i\sigma_I)$，$\sigma_I$ 为泡利矩阵，e_j^I 表示余三分量，物理坐标的指标用小写字母标记，内部 SU(2) 的指标用大写字母标记。在对应算符的显式作用中，B 包含的曲率项 O_F 被量子化为一个和乐算符构成的圈，沿 SNWF 所定义的基图已存在的边行进。由于和乐算符的表示可以与这些边关联的表示耦合为平凡表示，对应一条被湮灭的边，因此需要在物理哈密顿算符 $\hat{H}$ 中添加相应投影算符，以保证该算符确实保图 [50]。$Q^{jk}O_{D_j}O_{D_k}$ 在 A, E 变量下的贡献为

$$Q^{jk}O_{D_j}O_{D_k}(\sigma) = \frac{(O_E)^j_J (O_E)^k_K (O_F)^{J'}_{j\ell} (O_E)^\ell_{J'} (O_F)^{K'}_{k\ell'} (O_E)^{\ell'}_{K'} \delta^{JK}}{\det(O_E)}(\sigma) \tag{44}$$

$$= \frac{1}{4}\left[\mathrm{Tr}\,(B\tau_I)\right]\left[\mathrm{Tr}\,(B\tau_J)\right]\delta^{IJ}. \tag{45}$$

Therefore, the basic building block of the individual physical Hamiltonians is $\mathrm{Tr}\left(B\tau_\mu\right)$ and certain powers thereof, respectively. The quantization of $\mathrm{Tr}\left(B\tau_\mu\right)$ can be performed along the lines of how the Hamiltonian constraint is implemented in the context of the Dirac quantization. However, there exists one difference compared to the case of the Dirac quantization where the constraints but not a physical Hamiltonian are quantized. At the classical level, the physical Hamiltonian is a spatially diffeomorphism-invariant quantity, and one aims at defining the corresponding operator also with the same symmetry. As has been shown in [11], spatially diffeomorphism-invariant operators that are graph modifying do not exist in the LQG representation. Therefore, one needs to quantize the physical Hamiltonians in a graph-preserving manner. The physical Hilbert space based on the LQG representation involves SNWF defined on all graphs that can be embedded in the spatial manifold. Thus, graph-preserving or also called graph-nonchanging operators always come along with an infinite number of conservation laws, one for each graph, that are completely absent in the classical theory. Although such a graph-preserving property might be of advantage if one wants to use current semiclassical techniques as discussed already in section "(Extended) Master Constraint", here, an additional motivation to work in the AQG framework is to avoid these additional conservation laws in the quantum theory. Because all physical Hamiltonians are spatially diffeomorphism invariant by construction, they can be promoted to operators using the ITP Hilbert space AQG is based on and the usual quantization strategy for these operators. For the models displayed in (42), the usual embedded LQG and their corresponding AQG quantization exist [50, 56, 57]. As can be seen in (42), the contribution related to the spatial diffeomorphisms enters differently into the Hamiltonian density of the four scalar field model, but one can nevertheless quantize this quantity in the LQG and AQG representation as shown in [57].

因此，各个物理哈密顿量的基本构造块分别是 $\mathrm{Tr}\left(B\tau_\mu\right)$ 及其若干次幂。$\mathrm{Tr}\left(B\tau_\mu\right)$ 的量子化可以遵循狄拉克量子化框架下实现哈密顿约束的思路完成，但和狄拉克量子化 (该方案中量子化的对象是约束而非物理哈密顿量) 相比存在一处差异: 在经典层面，物理哈密顿量是空间微分同胚不变量，我们的目标是构造出同样具备该对称性的对应算符。正如文献 [11] 所示，在 LQG 表示中不存在改变图结构的空间微分同胚不变算符，因此物理哈密顿量必须以保图的方式量子化。基于 LQG 表示的物理希尔伯特空间包含定义在所有可嵌入空间流形的图上的 SNWF，因此保图 (也称为不变图) 算符总会附带无穷多个守恒律——每个图对应一个守恒律，这在经典理论中是完全不存在的。尽管在应用现有半经典技术时 (正如“(推广) 主约束”小节已讨论的)，这种保图性质可能存在优势，本文在 AQG 框架下工作的另一个动机是避免量子理论中出现这些额外的守恒律。由于所有物理哈密顿量按构造都是空间微分同胚不变的，它们可以借助 AQG 所基于的 ITP 希尔伯特空间，沿用这类算符的常规量子化策略提升为算符。对于式 (42) 列出的模型，常规嵌入 LQG 及其对应的 AQG 量子化都存在 [50, 56, 57]。从式 (42) 可以看到，空间微分同胚相关的贡献对四个标量场模型的哈密顿密度有不同形式的贡献，但正如文献 [57] 所示，该量仍然可以在 LQG 和 AQG 表示中完成量子化。

Reduced Quantization of Type II Models

II 类模型的约化量子化

Examples for quantum models of type II can be found in [39, 68, 111] that use either a massless scalar field, non-rotational dust, or a phantom field often used in k-essence, respectively, as a reference field for the Hamiltonian constraint. The corresponding physical Hamiltonians of these model are given by

II 类量子模型的例子可见 [39, 68, 111] ,这些例子分别采用无质量标量场、无旋转尘埃,或是 k-essence 理论中常用的 phantom 场作为哈密顿约束的参考场，对应的物理哈密顿量如下

$$H = \int_\sigma d^3x \sqrt{-\sqrt{\widetilde{Q}}\widetilde{O}_C + \sqrt{\widetilde{Q}}\sqrt{\left(\widetilde{O}_C\right)^2 - \widetilde{Q}^{ab}\left(\widetilde{O}_D\right)_a\left(\widetilde{O}_D\right)_b}}, \tag{46}$$

$$H = \int_\sigma d^3x\, \mathrm{sgn}\left(\widetilde{O}_C\right)\widetilde{O}_C \;\; [68]$$

$$H = \int_\sigma d^3x \sqrt{\frac{1}{2}\left[\left(\widetilde{O}_C\right)^2 - \widetilde{O}_{qDD} - \alpha^2\widetilde{Q}\right] + \sqrt{\frac{1}{4}\left[\left(\widetilde{O}_C\right)^2 - \widetilde{O}_{qDD} - \alpha^2\widetilde{Q}\right]^2 - \alpha^2\widetilde{O}_{qDD}Q}}$$

with $\widetilde{O}_{qDD} := \widetilde{Q}^{ab}\left(\widetilde{O}_D\right)_a\left(\widetilde{O}_D\right)_b$, [111]

其中 $\widetilde{O}_{qDD} := \widetilde{Q}^{ab}\left(\widetilde{O}_D\right)_a\left(\widetilde{O}_D\right)_b$, [111]

where α is a parameter involved in the Lagrangian of the model in [111] and a tilde is used on the top of all observables because these observables are only constructed with respect to the Hamiltonian constraint. Hence, the main difference to the type I models is that here the spatial diffeomorphism constraint needs to be solved in the quantum theory. As a consequence, the usual LQG representation cannot be used for the physical Hilbert space here. This has been circumvented in [68] by using the diffeomorphism-invariant Hilbert space $\mathcal{H}_{\text{diff}}$, usually constructed in the Dirac quantization approach, as the physical Hilbert space. An alternative is to work in the framework of AQG where these physical Hamiltonians can be quantized using the standard procedure for the individual contributions inside the (double) square roots following the strategy discussed in section ”Reduced Quantization of Type I Models”. The model in [111] was discussed at the classical level only so far in the literature but can in principle be quantized with the same techniques. Note that if one works on $\mathcal{H}_{\text{diff}}$, then one expects that the operator $\widehat{\widehat{O}}_{qDD}$ annihilates all states in this space; therefore, these contributions were neglected in [39] by hand. In this case, one has $H = \int d^3x \sqrt{-2\sqrt{\widetilde{Q}(x)\,\widetilde{O}_C}(x)}$, and this model can be understood as the full GR generalization of the APS model [13-15] in LQC where the inflaton is chosen as the clock and when the contribution of the potential is subdominant. In case one fully incorporates the potential in both models, then the final physical Hamiltonian becomes time dependent, and the usual issue of finding a suitable physical inner product in such situations is present. This can be avoided by coupling the reference matter in addition to the inflaton potential as has, for instance, been done in [53, 54, 59] with the price to pay that in the cosmological context, one needs to work with two-fluid models. As a first attempt in [57], a coupling of four Klein-Gordon scalar fields as reference fields was considered in order to define a corresponding type I model of the type II model in [39]. However, as shown in [57], such a model yields to a physical Hamiltonian that does contain the contribution of O_{D_j} not in the combination $Q^{jk}O_{D_j}O_{D_k}$ or individual components thereof but involves $\delta^{jk}O_{D_j}O_{D_k}$. As the discussion above showed in order to quantize these quantities in the LQG representation, the contraction with the inverse metric is crucial as otherwise the operator does not exist. Thus, if one restricts to the LQG representation, then for the model in [39] based on the Dirac quantization of the spatial diffeomorphism, the quantization program can be completed, whereas this is not possible by simply using ordinary Klein-Gordon scalar fields as reference fields in the reduced case. The latter issue can be avoided by adding three more fields to the system in a suitable manner so that the system also becomes second class. A reduction with respect to the second-class constraints yields a first-class system with the physical Hamiltonian shown in (42) for which the quantization program can be similarly completed. In this sense, a comparison of type I and type II models provides also some insights on the similarities and differences between Dirac and reduced quantization as far as the spatial diffeomorphisms are concerned. This, however, requires, if not working in the full theory, to consider symmetry reduced models where O_{D_j} does not trivially vanish, which goes beyond homogeneous and isotropic models.

其中 α 是文献 [111] 中该模型拉格朗日量包含的参数，且所有可观测量上方都加了波浪线，因为这些可观测量仅由哈密顿约束构造。因此，这类模型与 I 类模型的主要区别在于，此处需要在量子理论中求解空间微分同胚约束。由此导致，此处的物理希尔伯特空间无法使用常规圈量子引力 (LQG) 表示。文献 [68] 通过将狄拉克量子化方法中通常构造的微分同胚不变希尔伯特空间 $\mathcal{H}_{\text{diff}}$ 用作物理希尔伯特空间，规避了这一问题。另一种方法是在代数量子引力 (AQG) 框架下工作，遵循"I 类模型约化量子化"一节讨论的策略，对 (双) 根号内的单独贡献使用标准流程对这些物理哈密顿量进行量子化。文献 [111] 中的模型目前在文献中仅在经典层面得到讨论，但原则上可以用相同技术进行量子化。注意，如果在 $\mathcal{H}_{\text{diff}}$ 上工作，可以预期算符 $\widehat{\widetilde{O}}_{qDD}$ 湮灭该空间内的所有态；因此文献 [39] 手动忽略了这些贡献。在该情况下，可得 $H = \int d^3x\sqrt{-2\sqrt{\widetilde{Q}(x)\,\widetilde{O}_C}(x)}$，该模型可以理解为圈量子宇宙学 (LQC) 中 APS 模型 [13-15] 在完整广义相对论中的推广，其中将暴胀子选作时钟，且势的贡献处于次要地位。如果在两种模型中都完全纳入势，最终的物理哈密顿量会含时，此时就会出现这类情形下寻找合适物理内积的常规问题。可以通过额外引入参考物质与暴胀子势耦合来避免这一点，例如 [53, 54, 59] 中就采用了这种做法，代价是在宇宙学背景下需要使用双流体模型。文献 [57] 作为首次尝试，考虑将四个克莱因-戈登场耦合作为参考场，以便为文献 [39] 中的 II 类模型定义对应的 I 类模型。但正如文献 [57] 所示，该模型得到的物理哈密顿量中，O_{D_j} 的贡献并非出现在组合 $Q^{jk}O_{D_j}O_{D_k}$ 或其单独分量中，而是包含 $\delta^{jk}O_{D_j}O_{D_k}$。正如前文讨论所示，为了在 LQG 表示中对这些量量子化，与逆度量缩并至关重要，否则该算符不存在。因此，如果限制在 LQG 表示内，文献 [39] 中基于空间微分同胚狄拉克量子化的模型可以完成量子化方案，而在约化情形下仅使用常规克莱因-戈登标量场作为参考场则无法实现。后者的问题可以通过向系统中合适地额外加入三个场来避免，这样系统也会成为第二类约束系统。对第二类约束进行约化后会得到第一类系统，其物理哈密顿量如 (42) 式所示，同样可以完成量子化方案。就此而言，对比 I 类和 II 类模型也能就空间微分同胚问题，帮助我们理解狄拉克量子化和约化量子化的异同。但这一点要求，若非在完整理论中工作，则需要考虑 O_{D_j} 不平凡为零的对称约化模型，这已经超出了均匀各向同性模型的范畴。

Summary and Outlook

总结与展望

Our exposition of the status of the dynamics of LQG suggests the following:

我们对圈量子引力动力学研究现状的阐述可以总结如下:

1. Despite some progress, the quantization before constraining route is still far away from making contact to phenomenology. Even after the issue of anomalies has been settled in the full theory for Lorentzian signature including all observed matter, one would still need to compute the kernel of the constraints, equip it with an inner product, and construct operator representations of Dirac observables thereon. On the other hand, the techniques developed in this route are employed almost without change in the reduced phase space approach and are thus of outmost importance.

1. 尽管已取得部分进展，约束前量子化路径仍远未与唯象学建立联系。即使在包含所有已观测物质的洛伦兹号差 full 理论中解决了反常问题，仍需要计算约束的核、为其赋予内积，并在其上构造狄拉克可观测量的算符表示。另一方面，该路径发展出的技术几乎无需修改即可直接应用于约化相空间方法，因此具有极为重要的意义。

2. The quantization after constraining route in that sense is much more economic and sidesteps all of these complications. While to be practically useful it requires to use matter to extract the reduced phase space, this is not a disadvantage in view of the fact that a universe without matter is unphysical; thus, one may as well use it to make progress and also to move much closer to phenomenology.

2. 从这个角度来看，约束后量子化路径要简洁得多，绕开了所有这些复杂问题。尽管它在实际应用中需要利用物质提取约化相空间，但考虑到没有物质的宇宙本身就是非物理的，这并不算缺点；因此我们完全可以利用该路径推进研究，也能更快地靠近唯象学。

3. There is one technical issue common to both routes, which is the presence of quantization ambiguities. These have to be downsized to a finite number to render the theory predictive. Renormalization is a possible avenue to reach this goal.

3. 两条路径存在一个共同的技术问题: 量子化歧义的存在。必须将其缩减到有限数量才能让理论具备预言能力，重整化是实现这一目标的可能途径。

Cross-References

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